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More information on the Dynamic Number Project, and additional Geometric Functions activities, are available here: www.kcptech.com/dynamicnumber

## GEOMETRIC FUNCTIONS

This is one of a series of activities in which students explore functions geometrically. Because both the independent and dependent variables of a geometric function are points, students can vary the independent variable by dragging it and observing directly the behavior of the dependent variable. By tracing both variable points, the relationship between them becomes visible as an image on the screen. Students explore domain, range, composition, and inverse through direct manipulation of variables, and these concepts manifest themselves through visual images that reveal their fundamental aspects.

Many students may have met functions only in the numeric realm, and may not be aware as they begin these activities that function, transformation, and mapping are mathematical synonyms. It's strongly recommended that students begin with these two activities (Indentify Functions and Indentify Function Families) to begin to get comfortable with functions in which the variables are points, developing some practice in manipulating the independent variable while observing the behavior of the dependent variable. Class discussions of the similarities and differences between algebraic and geometric functions help students see that these geometric relationships really do behave as functions, and encourage students' awareness of the continuous variability of both geometric and numeric variables. Students come to understand how functions in both realms express relationships between independent and dependent variables.

By engaging with geometric functions and discussing their similarities and differences with numeric functions, students are spurred to go beyond initial simplistic ideas, such as the common belief the essential nature of a function is an algebraic representation in the form $y=2 x$. As they use geometric functions to study domain, range, composition, and related topics, students have an opportunity to work with geometric representations of these concepts, to relate them to algebraic representations, and thus to develop a deeper and more sophisticated understanding of functions.

## ACTIVITY OVERVIEW

This activity introduces the ideas of relation and function, using geometric points as the independent and dependent variables. Students explore and describe relationships between points, and use examples and non-examples to formulate a definition of function.

Ideally this activity is students' formal introduction to the definition of a function. If students are already familiar with functions in the numeric realm, this activity is an excellent way to review the concept using a completely different representation, to spur students' thinking about the similarities between these two ways of exploring functions, and to stimulate students to generalize and move toward a more abstract understanding of the concept.

After completing this activity, students should be able to:

- Drag independent points and observe and describe the behavior of dependent point(s).
- Think of points as variables that may be independent, or may be dependent on other variables.
- Use the drag test to distinguish relations in which the dependent point has a single location from those in which it can have two or more locations.
- Formulate and discuss their own definition of function.
- Use their definition to categorize relationships between variable points as functions or not functions.
- Become familiar with use of the term relation to describe two variables that have a relationship that may or may not be a function.
- Create their own constructions as examples of functions and non-functions (optional).

Several of these objectives are particularly important for the activities that follow. Drag independent points: Further geometric explorations of function make consistent use of points as the independent and dependent variables. The ease of dragging the independent variable to observe the behavior of the function is a main advantage of this approach; this activity prepares students for further explorations.

The drag test: The drag test is students' main tool for exposing behavior of relations and for identifying and categorizing functions, precisely because it provides students with the ability to vary the independent variable.

Defining and identifying functions: The discourse that students engage in, both with their partners and in the whole-class summary discussion, is critical to their building a solid concept of function. During the group discussion it's advantageous, when possible, to refer back to specific pages of the sketch, and specific behavior of functions and non-functions, to help this cognitive development take place.

On your own: Some students will enjoy constructing their own functions and nonfunctions, and challenging their classmates to guess which is which before dragging.

This will be easier if your students already have some experience with creating Sketchpad constructions, but page 12 gives fairly explicit step-by-step instructions that many students will be able to follow, even without extensive construction experience. Later activities will focus on such constructions.

## VOCABULARY

Independent point and dependent point: The activity uses these interchangeably with independent variable and dependent variable. Using a common vocabulary, in which variables may be numbers or may be points, makes it easier for students to connect the functions they meet in a geometric context with the functions they meet in a numeric context.

Relation: The activity uses the term relation to denote the relationship between points. There's no need right now to develop a precise definition of relation. It's enough to tell students that they are exploring relations in this activity and that the word has a more precise mathematical meaning that they'll learn later. (It's difficult to provide non-examples of relations, making this term hard to define.)

Function: One objective of this activity is to agree on a definition for function. This should be a focus of the closing class discussion. During that discussion you should avoid imposing a particular definition, but instead strive for clarity and consensus on a definition generated by students. The class will refine this preliminary definition as they engage in later activities in this unit.

## INTRODUCE AND MODEL

Use a projector to show sketch page 1, and ask a student to read the directions. Call another student to the interactive whiteboard (if you have one) or to the computer to drag point $a$. Ask for a description of the result, and as a class agree that $a$ and $y$ are related, that $a$ is the independent variable, and that $y$ is the dependent variable. If students are already familiar with these terms, take a moment to discuss the relationship between numeric and geometric variables. Tell students to fill in the first cell of the Q1 grid on their worksheets to show $a \rightarrow y$.

Do enough dragging to see that $a$ and $y$ can be dragged to the same location, and tell students that they should include this fact in their description of the relationship. Have the student try to drag point $y$, and note that on this page you cannot drag the dependent variable. Have students complete the first row of the table by writing a sentence describing what they notice about the relationship.

Have the same student try to drag $b$; the class should agree that $b$ might also be a dependent variable.

Go to page 2 and have another student read the directions. Tell the class that their job is going to be to figure out their own definition of a function. Point out that the
directions say that $x$ and $x^{\prime}$ are an example of a function, and have a fourth student drag $x$ while the class observes the result. Then have the student drag $y$ to observe the behavior of a non-example. Tell the class that they'll have a number of examples and non-examples, and that their job is to work with their partner to describe the behavior of the various functions and non-functions and to agree on, and write down, the clearest definition of a function that their team can formulate.

Assign students to partners, send them in pairs to the computers, and ask them to finish identifying independent and dependent variables on page 1 , and then work through pages 2 through 11 . On each page they should describe the behavior of the function and the non-function, and in the process figure out the clearest way to describe the difference and define a function. As you circulate, observe how students interact with and write about the features that appear on various pages. (Some of the particularly important features are mentioned in the bulleted list in the Discuss and Summarize notes below.) Ask students probing questions about their observations and descriptions, and make mental notes of items you want to address, and students you want to call on, during the class discussion.

Tell students that pages 11 and 12 are optional, but are a nice challenge if they get finished with the earlier pages, have written their formal definition, and have shared it with you.

As you circulate, make sure students are writing clear descriptions, in complete sentences, of the behavior they observe. Students should observe when particular locations of the independent variable are required to produce multiple dependent variables. If students initially think that both relations on a given page are functions, tell them to drag the independent variables to more locations. As students finish their definitions, have them type the definition on your projection computer, with the projector turned off, so that you can easily display them later.

## DISCUSS AND SUMMARIZE

When students have had enough time to investigate and describe the examples and non-examples on pages 1 through 11, call them together for a class discussion. (Always have monitors turned off or laptop lids closed for such discussions.)

Begin by asking students which pages they found most interesting. Here are some observations that students might make about the various pages:

- On page 1 , one of the independent variables has no dependent variable, and one has two dependent variables.
- Several pages have arrows connecting independent and dependent variables. Ask students what they thought the arrows meant. (The arrows are a graphic
embellishment intended to make the functional dependence seem more concrete; it's important that students not attribute inappropriate mathematical significance to them.)
- On page 4 , the dependent variable seems to explode into 5 different variables when you drag, creating an interesting effect.
- On page 5 , point $b$ ' never moves, so students may want to say that it's not a relation. This can generate a nice discussion about the definition. One possible question is this: "Try writing the definition both ways, including and excluding this case. Which definition is simpler?"
- On pages 6 and 7 , students have to drag the independent variables quite far before the dependent variable splits into two. Students may wonder whether that means these relations are sometimes functions and sometimes not functions. A good response might be to observe that there may be times when we can take advantage of such behavior, treating a relation like a function when that's the way it behaves. In a real-life example, we may suppose that all shadows point in the same direction, but when the objects casting those shadows are separated by thousands of miles, we must refine that statement.
- On page 8, the dependent variables move discretely rather than continuously. Consider asking students what is different about the behavior on this page, and encourage a variety of descriptions of the discontinuous behavior. You might ask students to compare this function to the constant-function behavior of $b$ and $b$ ' on page 5. (The function on page 8 behaves like a constant function when you drag the independent variable a short distance - but as you continue the dependent variable suddenly jumps.
- Concerning page 9, ask students whether the arrows made any difference. Did they press the Hide Arrows button? Did the points behave differently afterward?
- On page 10 , dragging $t$ has no effect on dependent variable $t^{\prime}$. Students may question whether this is a function at all, since they see no relationship. This can lead to a nice discussion of whether one should exclude one particular kind of behavior (not moving) from the behaviors of interest. Dealing with this question now will help students understand that constant functions are indeed functions.
- On page 11, the periodic behavior of both relations may catch students' interest, particularly because if they were to trace the dependent variables, both would trace out the shape of a circle, one by tracing the entire circle, and the other by tracing out each branch separately.
(If students are already familiar with the vertical line test, they may bring the
question up now, because the trace of $n$ ' would not pass the vertical line test. This is an opportunity to stress that the vertical line test is specific to a particular representation, numeric values plotted on a Cartesian graph, and to emphasize that the real test is not whether a vertical line intersects a Cartesian graph, but whether two or more different values of the dependent variable exist for a single value of the independent variable.)

Ask students to identify common threads they observed as they explored examples and non-examples of functions. Possible observations:

- Students may observe that the non-functions were more interesting than the functions.
- Students will almost certainly observe that the functions always have a single output variable; never more than one and never less than one.
- Students may ask about a function that sometimes has one output value, and sometimes no output values. The definition usually declares that there's exactly one output value, so functions like this may prompt mathematicians to take extreme measures like restricting the domain (Beware! Here be dragons!) or inventing new kinds of numbers (Imaginary numbers! No joke!).

Display definitions from various teams. Encourage students to discuss and question the various definitions, and ask probing questions of your own to try to clarify students' thinking and to zero in on the critical aspects of their definitions. Discuss, and list, criteria for a good definition. Formulate one or perhaps two class definitions that all can agree on, ideally drawing ideas from a number of the proposed definitions. Have students copy the definition(s) to their journals. Students haven't been asked to define relation at this time, and haven't seen non-examples. What do they think a relation is?

A useful final question if there's time: Why do you think mathematicians consider it so important to study functions?

Q1

| Independent <br> Point | Dependent <br> Point $(\mathbf{s})$ |
| :---: | :--- | | Relationship |
| :--- |
| $a \rightarrow y$ | | Point $y$ is a reflection of $a$ in a hidden |
| :--- |
| line. | | $z \rightarrow b$ | Point $b$ is a $180^{\circ}$ rotation of $z$ about a <br> hidden point. |
| :---: | :--- |
| $c \rightarrow d$ | Point $d$ is a translation of $c$. |
| $v \rightarrow$ | Point $v$ has no relationship to any other <br> point. |
| $w \rightarrow e \& x$ | Points $e$ and $x$ are dilations of $w$ with |


|  | respect to a hidden point. The ratios of <br> dilation are not equal. |
| :--- | :--- |

Q2 There is only one $x^{\prime}$, but there are two $y^{\prime}$ 's.
Q3 There is only one $b^{\prime}$, but there are two $a^{\prime}$ s.
Q4 On pages 4 through 11, list the function and the non-function. For each page, write an observation or question you have about the relationships.

| Page | Function | Non- <br> function | Observations and Questions |
| :---: | :--- | :--- | :--- |
| 4 | $q \rightarrow q^{\prime}$ | $p \rightarrow p^{\prime}$ | There are multiple $p^{\prime}$ 's. |
| 5 | $b \rightarrow b^{\prime}$ | $a \rightarrow a^{\prime}$ | There are multiple $a^{\prime}$ s. |
| 6 | $c \rightarrow c^{\prime}$ | $d \rightarrow d^{\prime}$ | Sometimes there is one $d^{\prime}$, sometimes two. |
| 7 | $v \rightarrow v^{\prime}$ | $u \rightarrow u^{\prime}$ | Sometimes there is one $u^{\prime}$, sometimes three. |
| 8 | $s \rightarrow s^{\prime}$ | $r \rightarrow r^{\prime}$ | Both $r^{\prime}$ and $s^{\prime}$ move in a jerky way. <br> Sometimes there is one $r^{\prime}$, sometimes three. |
| 9 | $w \rightarrow w^{\prime}$ | $z \rightarrow z^{\prime}$ | Sometimes there is one $z^{\prime}$, sometimes two. |
| 10 | $t \rightarrow t^{\prime}$ | $e \rightarrow e^{\prime}$ | There are two $e^{\prime}$ 's. |
| 11 | $n \rightarrow n^{\prime}$ | $m \rightarrow m^{\prime}$ | Both $m^{\prime}$ and $n^{\prime}$ move in circles, but there are <br> two $m^{\prime} s$, and only one $n^{\prime}$. |

Q5 In a function, for every location of the independent variable there is exactly one location for the dependent variable.

## LIST OF TRANSFORMATIONS

This list summarizes the transformations on each page of Identify Functions.gsp.

| Page | Function | Behavior |
| :---: | :---: | :--- |
| 1 | $a \rightarrow y$ | reflection |
| 1 | $c \rightarrow d$ | translation |
| 1 | $w \rightarrow e, x$ | dilation by scale factor of 2, translation |
| 1 | $v$ | none |
| 1 | $z \rightarrow b$ | rotation by $180^{\circ}$ or dilation by a scale factor of -1 |
| 2 | $x \rightarrow x^{\prime}$ | reflection across horizontal mirror |
| 2 | $y \rightarrow y^{\prime}$ | reflection \& rotation |
| 3 | $a \rightarrow a^{\prime}$ | translation \& dilation |
| 3 | $b \rightarrow b^{\prime}$ | translation |
| 4 | $p \rightarrow p^{\prime}$ | a reflection \& four rotations |
| 4 | $q \rightarrow q^{\prime}$ | rotation by 180 or dilation by a scale factor of -1 |
| 5 | $a \rightarrow a^{\prime}$ | three different transformations |
| 5 | $b \rightarrow b^{\prime}$ | constant transformation |
| 6 | $c \rightarrow c^{\prime}$ | glide reflection |
| 6 | $d \rightarrow d^{\prime}$ | glide reflection \& translation |
| 7 | $u \rightarrow u^{\prime}$ | three transformations, two of which appear only when $u$ is on the <br> right half of the screen |
| 7 | $v \rightarrow v^{\prime}$ | translation |
| 8 | $r \rightarrow r^{\prime}$ | two different discrete transformations |
| 8 | $s \rightarrow s^{\prime}$ | discrete transformation |
| 9 | $w \rightarrow w^{\prime}$ | one transformation |
| 9 | $z \rightarrow z^{\prime}$ | two transformations, one of which appears only when $z$ is near <br> the bottom of the screen |
| 10 | $e \rightarrow e^{\prime}$ | two dilations |


| 10 | $t \rightarrow t^{\prime}$ | constant transformation |
| :---: | :---: | :--- |
| 11 | $m \rightarrow m^{\prime}$ | two transformations that together trace out a circle |
| 11 | $n \rightarrow n^{\prime}$ | one transformations that traces out a circle |

