

Chapter 2: Exploring Quadrilaterals

Summary: In this chapter students examine definitions and explore properties of special quadrilaterals. Methods of classifying quadrilaterals are investigated and the van Hiele levels of geometric thinking are introduced and the roles of technology in exploring, conjecturing and proving are highlighted.

Objectives:

Mathematical: Teachers will be able to

- Create definitions from examples and non-examples of geometrical figures.
- Understand the difference between partitional and hierarchical definitions.
- Critique definitions and select definitions that will be most useful.
- Use properties of quadrilaterals to create different hierarchical classifications.
- Construct proofs of mathematical statements or find counterexamples for mathematical statements that are false.

Technological: Teachers will be able to

- Measure segments and angles
- Use the calculator tool to explore relationships
- Create scripts/tools to construct complex geometrical figures
- Consider how technology can assist students in communicating with each other about geometrical ideas.
- Become familiar with different methods of dragging students may use when engaging in a mathematical exploration.

Pedagogical:

- Create activities that allow students to generate their own definitions.
- Analyze a student's van Hiele level of geometric thinking.
- Create tasks appropriate for students at different van Hiele levels.
- Create open-middled or open-ended geometry tasks for students.

Prerequisites:

Vocabulary: Technology Files:

Emergency Technology Files:

Required Materials:

Chapter 2: Exploring Quadrilaterals

Section 1: Defining in Geometry

In a typical geometry class, a teacher records a definition on the board and students dutifully copy it into their notebooks. Then, the teacher may provide examples of the object that is being defined. However, this order of presentation of definitions and examples can be reversed. Suppose a teacher started the class by providing students with examples and non-examples of a particular geometrical object and asked students to create a definition such that it encompassed all of the examples and excluded the non-examples. This reversal of activities involves students in the important mathematical process of creating definitions.

To illustrate this approach, we will consider the geometrical definition of a kite. Open the sketch “DefiningKites.gsp.” On the first page there is a construction of a kite that will always be a kite when any vertex of the kite is dragged. The second page of the sketch includes three different constructions of polygons that are never or are not always kites when a vertex of the polygon is dragged.

Tech Tip:

ENGAGING WITH CONTENT

Q1. Explore the example and non-examples of kites. Consider properties of sides, angles, and diagonals. Create a definition of a kite for which all examples of kites satisfy it and the non-examples do not.

Q2. Reflect on your thinking processes. What properties did you consider important or not important? Did you create definitions that did not work? How did you use the technology file? Were there other non-examples that you used to help create your definition?

CONSIDERATIONS FOR TEACHING

Q3. Describe benefits and drawbacks of allowing students to generate their own definitions versus a teacher providing a formal definition to students.

Q4. When students are generating their own definitions, describe how a teacher can bring these different ideas together so that as a class they are working from a single definition.

As you were creating your definition of a kite there may have been a number of properties that you were focusing on, but it is unlikely that you included all of these properties in your definition. There are differences between properties of an object and properties needed to define it. For example, properties of a square include: four sides, opposite sides congruent, opposite sides parallel, four right angles, diagonals

that are congruent, perpendicular and bisect each other. However, to define a square it is not necessary to include all of these properties in the statement of its definition. Rather, a definition focuses on those properties that are essential to the figure and are independent of each other. That is, one property is not derivable from the other.

When a teacher conducts a definition creating activity with a class of students, it is likely that individuals will arrive at different definitions. It is not uncommon to have different definitions for the same mathematical concept or object. For example, a brief search on the internet resulted in five different definitions of a square that are listed below.

Definition 1: A square is a quadrilateral with four congruent sides and four right angles.

Definition 2: A square is a parallelogram with two adjacent congruent sides and one right angle.

Definition 3: A square is a four-sided regular polygon.

Definition 4: A square is a rectangle with four congruent sides.

Definition 5: A square is a rhombus with one right angle.

Is it possible to have five different correct definitions? What makes one definition more useful than another? The answer to the first question is, “yes.” As long as the definitions apply to the same class of figures then they are considered equivalent. But if they result in two different classes of figures, then they are not equivalent. The answer to the second question is a bit more complex. Several authors have written about different criteria that can be used to determine whether a definition is useful (Edwards & Ward, 2008, p. 224). These criteria are listed below:

Hierarchy: A new concept is defined as a subset of a more general concept.

Existence: At least one instance of the defined concept exists.

Equivalence: Multiple definitions of the same concept are equivalent.

Acclimatization: The definition fits within an already existing deductive system.

Other characteristics of definitions that are often desired, but not required include:

elegance and **minimality**. The definition should be stated in the fewest number of words and/or symbols (elegance) and only include properties necessary (minimality).

Tech Tip:

ENGAGING WITH CONTENT

Q5. Determine if the five definitions of a square listed above are all equivalent. Explain.

Q6. Based on the stated criteria for mathematical definitions, which of the five definitions do you believe is the best? Explain.

Q7. Find or create a sixth definition of a square that is different from those that are listed. Explain why your definition is equivalent.

CONSIDERATIONS FOR TEACHING

Q8. Consider the five definitions of a square listed above. If you wanted to select one of the five to present to students which would you select? Why?

Q9. Of the five definitions of a square, which do you think would be most difficult to understand for students? Explain.

Q10. Consider the following definition of a quadrilateral: “A quadrilateral is a four-sided polygon.” What do students need to understand in order to make sense of this definition? Create three examples of quadrilaterals and three non-examples.

Section 2: Constructing and Exploring Properties of Special Quadrilaterals

A quadrilateral is a four-sided polygon. There are many different types of quadrilaterals, some of which have special classifications. These include: parallelograms, trapezoids, isosceles trapezoids, rectangles, rhombi, squares, and kites. Each type of quadrilateral is determined by key defining features.

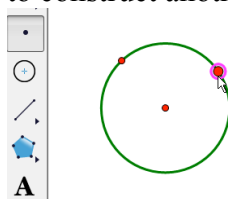
As we noted in Section 1, geometrical objects can be defined in multiple ways. These different definitions can have implications for the way in which they are constructed in a DGE. Consider, for example, the following two definitions of a square.


Definition 1: A square is a quadrilateral with four congruent sides and four right angles.

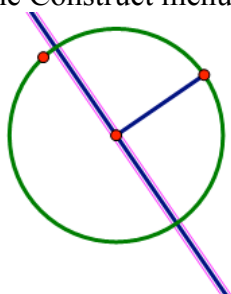
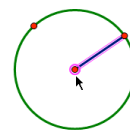
Definition 2: A square is a parallelogram with two adjacent congruent sides and one right angle.

Possible construction using Definition 1:

1. Open a new sketch and use the circle tool to create a circle. Use the point tool to construct another point on the circle.

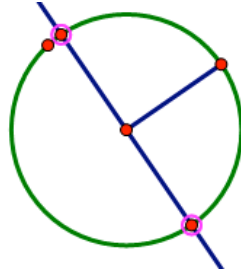


2. Use the selector tool  to select the newly constructed point and the center of the circle. Then, press Ctrl+L to connect with a segment. Notice the segment is “highlighted.” This segment will be one side of the square. With the segment highlighted, click on the center point. Choose Perpendicular Line from the Construct menu. Notice the line is now “highlighted.”

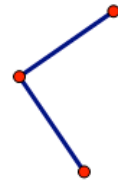


3. Construct the intersection of the perpendicular line and the circle. To do this, click on the circle so that both the circle and the line are “highlighted.” Then, choose Intersections from the Construct menu. Note that Sketchpad

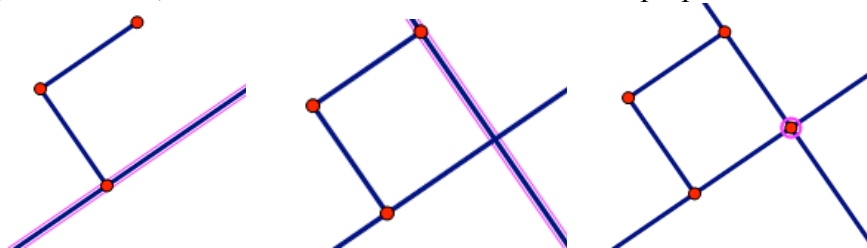
recognizes that there will be two points of intersection. We will only use one for this construction.



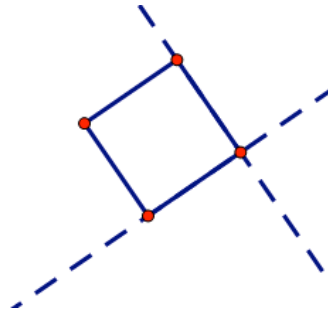
4. Use the selector tool to select the center of the circle and one of the points of intersection, then press Ctrl+L to connect the points with a segment. Notice the line becomes dotted and the segment becomes solid in color. So far we have created two segments that are congruent as they are radii of the same circle and we have constructed them such that they intersect at a right angle. But, the goal is to construct a square. Sometimes, constructions can get messy. Let's hide the pieces of this construction that we no longer need. Use the selector tool to select all items except the two congruent segments and their endpoints. Then choose Hide Objects from the Display menu.



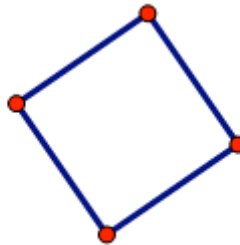
5. To create the third side of the square, use the selector tool to select one segment and the endpoint that is on the (hidden) circle. Then choose Perpendicular Line from the Construct menu. Repeat this step with the other segment. Then, construct the intersection of the two perpendicular lines.



6. Connect this intersection point with one of the endpoints used to create the perpendicular lines by selecting it and pressing Ctrl+L. Repeat this step with the other endpoint. Notice the perpendicular lines become dotted and the segments are solid in color.

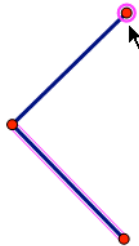


7. Finally, hide the perpendicular lines by selecting them and choosing Hide Perpendicular Lines from the Display menu.

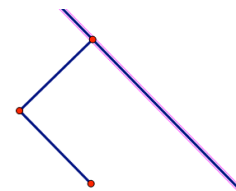


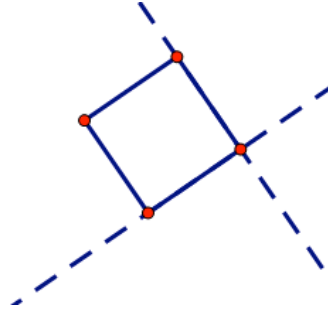
Possible construction using Definition 2:

1. Follow steps 1-4 from the previous construction to create two sides of the square.
2. Use the selector tool to select one segment and the endpoint of the other segment (see image below).

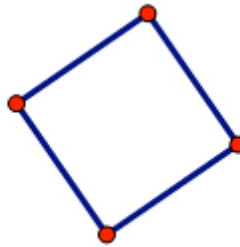


3. Choose Parallel Line from the Construct menu.
4. Repeat steps 2-3 above to create the fourth side of the square.
5. Use the selector tool to select the two lines created. Then, choose Intersection from the Construct menu.
6. Connect this intersection point with one of the endpoints used to create the parallel lines by selecting it and pressing Ctrl+L. Repeat this step with the other endpoint. Notice the lines become dotted and the segments are solid in color.






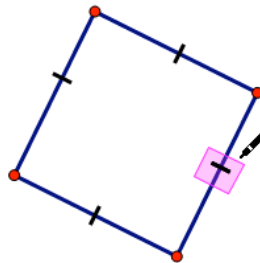
7. Finally, hide the lines by selecting them and choosing Hide Parallel Lines from the Display menu. Note that Sketchpad remembers how the lines were created. The lines are not parallel to each other, but they were created as parallel lines to other parts of the construction.



*****TECH TIP*****

We often use congruent markings in geometry. This can easily be done in GSP as well. To show congruent markings in the square above, start by clicking on the

marker tool, . Then click on each of the sides of the square.



Tech Tip:

ENGAGING WITH CONTENT

Q11. Select one of the two definitions and describe how you would construct a square using this definition.

Q12. A definition does not explicitly state every property of the figure it is defining. Explore the construction of your square to describe additional properties (for example, diagonals, symmetry, measures of angles). Record your properties below in Table 1.

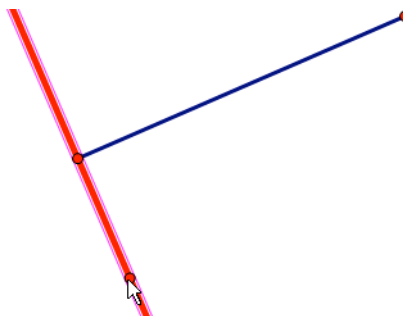
CONSIDERATIONS FOR TEACHING

Q13. Analyze the two definitions. Which properties of a square are highlighted? Is one of the definitions easier to use for constructing?

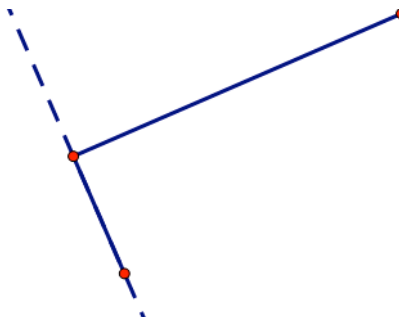
Open up the sketch, “Quads.gsp” or use your DGE to construct a square, rectangle, rhombus, kite, parallelogram, cyclic quadrilateral (a quadrilateral inscribed in a circle), and a trapezoid.

Possible construction of a Rectangle:

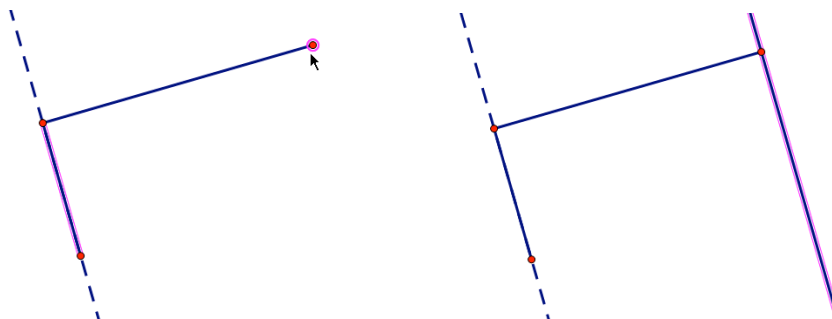
1. Open a new sketch. Use the Line tool to create a line segment.
2. Use the selector tool to select the line and a point on the line. Choose Perpendicular Line from the Construct menu. Next, use the point tool to place a point on the line.



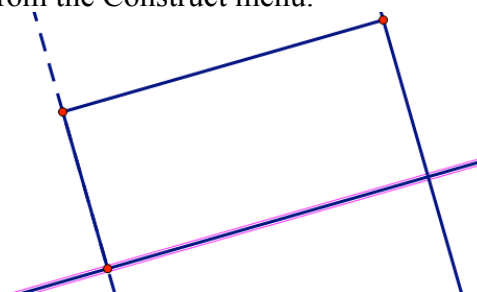
3. Connect this newly created point with the endpoint of the segment by pressing Ctrl+L. Notice the perpendicular line becomes dotted and the segment is solid in color.



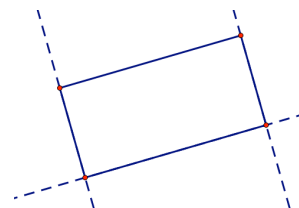
4. Use the selector tool to select the new segment and the unused endpoint of the original segment. Then, choose Parallel Line from the Construct menu.



5. With the parallel line highlighted, select the point created in step 2. Choose Perpendicular Line from the Construct menu.

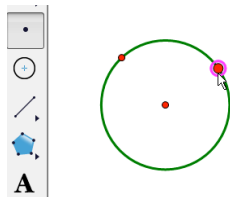


6. With this new perpendicular line “highlighted,” select the parallel line created in step 4. Then, choose Intersection from the Construct menu.
7. Connect this intersection point with one of the adjacent endpoints by selecting it and pressing Ctrl+L. Repeat this step with the other endpoint. Notice the lines become dotted and the segments are solid in color.
8. Finally, hide the lines by selecting them and choosing Hide Lines from the Display menu.

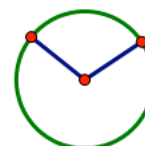


Possible construction of a Rhombus:

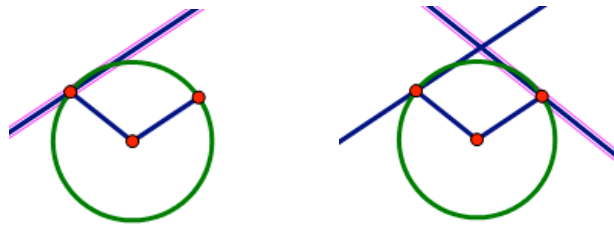
1. Open a new sketch and use the circle tool to create a circle. Use the point tool to construct another point on the circle.



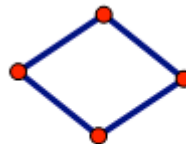
2. Construct a segment from each of the points on the circle to the center point by pressing Ctrl+L.



3. Use the selector tool to select one of the segments and the endpoint of the segment not selected. Then, choose Parallel Line from the Construct menu. Repeat this step with the other segment.

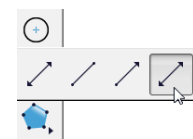


4. With the new parallel line “highlighted,” select the first parallel line constructed and choose Intersection from the Construct menu.
5. Connect this intersection point with one of the adjacent endpoints by selecting it and pressing Ctrl+L. Repeat this step with the other endpoint. Notice the lines become dotted and the segments are solid in color.
6. Finally, hide the lines and the circle by selecting them and choosing Hide Path Objects from the Display menu.



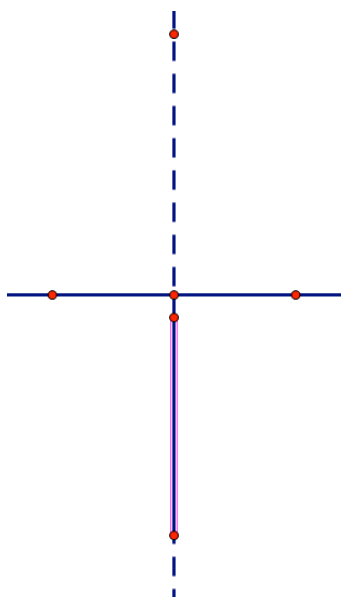
Possible construction of a Kite:

1. Open a new sketch and begin by selecting and creating a vertical line using the line tool. Two points will automatically be constructed on that line.
2. Use the selector tool to select the lower point and the line. Then, choose Perpendicular Line from the Construct menu.
3. Use the point tool to construct a point on the left side of the newly created perpendicular line. Simply click on the point tool and move the mouse until the point lies on the line (the line will turn red in color) and click again to place the point on the line.
4. Use the selector tool and double click on the original line. This action marks that line as a Mirror Line. Notice that the line “explodes.” Then, with the point constructed on the left side still highlighted, choose Reflect from the Transform menu.

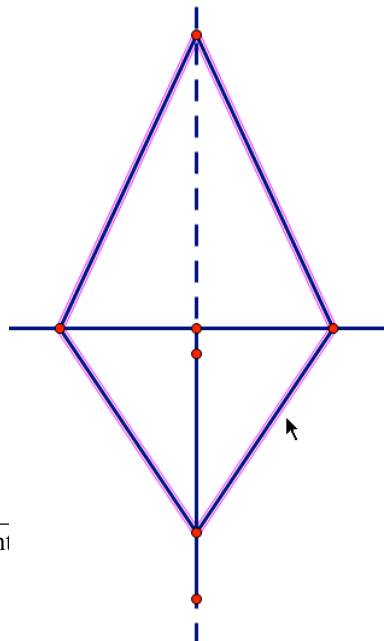


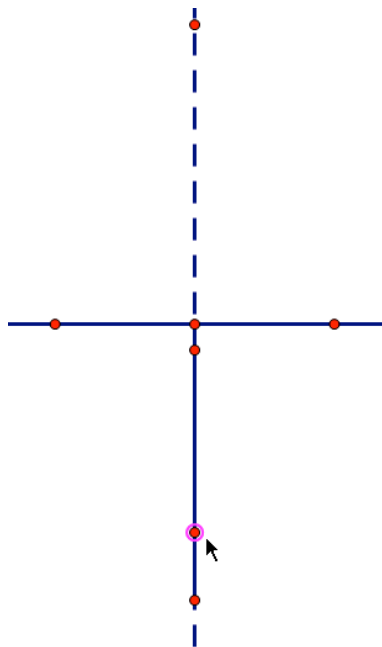
5. The final vertex of the kite can be tricky to construct. If a fourth point is simply constructed on the vertical line, the “kite” could collapse into a concave figure. To avoid this, we will create a point that is *secretly* restricted. Here’s how:

- a) Construct two points below the perpendicular line constructed in step 2 above.
- b) Construct a line segment between the new points (see picture below).



- c) Next, construct a point on this new segment. Then, construct segments for the four sides of the kite.

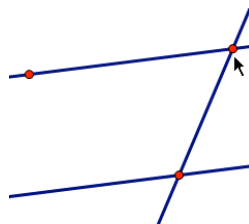
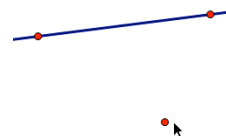




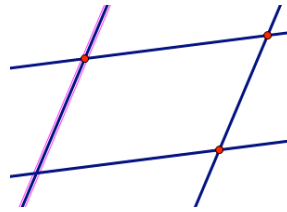
- d) Finally, use the selector tool to select the two lines, segment, and three unused points of the kite and choose Hide Objects from the display menu.

Possible construction of a Parallelogram:

1. Open a new sketch and construct a line and a point not on the line.
2. Use the selector tool and select the line and the point not on the line. Then, choose Parallel Line from the Construct menu.
3. Construct a line between the point constructed in step 1 and one of the points on the original line.



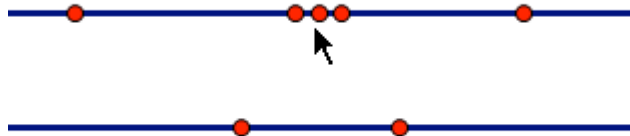
4. Use the selector tool and select this new line (from step 3) and the other point from the original line. Then, choose Parallel Line from the Construct menu.



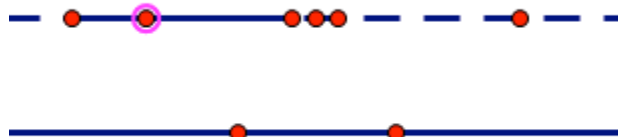
5. With this line still highlighted, select the first parallel line. Then, choose Intersection from the Construct menu.
6. Select the four intersection points and connect them with constructed line segments by pressing Ctrl+L. Notice the lines become dotted and the segments become solid in color. Finally, select and hide the four lines.

Possible construction of a Trapezoid:

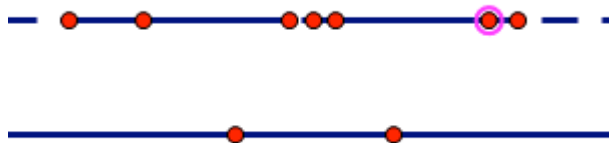
1. Open a new sketch and construct a line and a point not on the line.
2. Use the selector tool to select the line and point (from step 1) and choose Parallel Line from the Construct menu.
3. Next, construct four points on the parallel line – two to the left of the point from step 1 and two on the right (see picture below).



4. Use the selector tool to select the two left most points and construct a line segment by pressing Ctrl+L. Notice the segment becomes solid in color while the parallel line becomes dotted. With the segment still highlighted, select Point on Segment from the Construct menu.



5. Repeat step 4 with the two right most points.



6. Now, select the two constructed points (from steps 4 and 5) and the two points on the original line. Connect them with constructed line segments by pressing Ctrl+L.



7. Finally, clean up the construction by hiding all lines, segments, and points that do not need to be seen.



Use the sketch to identify different properties. Record them in table 1. For example, a property of the square might be that the opposite sides are always parallel. After you have listed the properties, construct a definition.

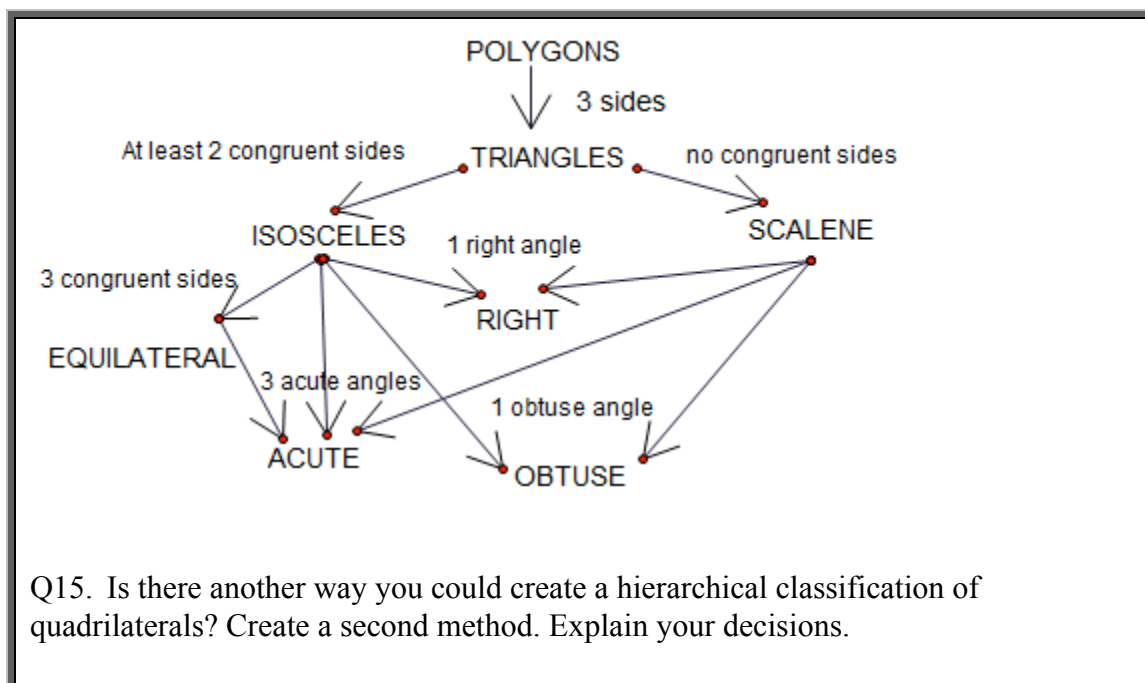
Properties	Sides	Angles	Diagonals	Definition
Cyclic Quadrilateral				
Parallelogram				
Rhombus				
Square				
Kite				
Trapezoid				
Rectangle				

Table 1. A list of properties of different quadrilaterals and their definitions

Tech Tip:

ENGAGING WITH CONTENT

Q14. Consider the list of properties that you created in Table 1. Use this list to create a hierarchical classification of quadrilaterals that includes a square, rectangle, rhombus, kite, parallelogram, cyclic quadrilateral, and a trapezoid. Be prepared to explain how you decided to create your chart. See an example of a hierarchical classification of triangles below.



You might have noticed that the trapezoid that was constructed in the DGE could be dragged to become a parallelogram. If you look in different geometry textbooks or conduct a search on the internet you may find at least two different definitions of a trapezoid. Consider the following two.

Definition 1. A trapezoid is a quadrilateral with exactly one pair of parallel sides.

Definition 2. A trapezoid is a quadrilateral with at least one pair of parallel sides.

ENGAGING WITH CONTENT

Q16. Is a parallelogram a trapezoid using definition 1? What about definition 2? Explain.

Q17. Which definition of trapezoid do you prefer? Why?

Notice that the two definitions of a trapezoid are not equivalent because they result in different classes of figures. In the previous section you selected a preferred definition of square that was equivalent to the other definitions of square. However, in selecting a preferred definition of trapezoid in this case you are actually choosing a preferred hierarchical classification system. These differences are both important and also very subtle.

CONSIDERATIONS FOR TEACHING

Q18. What are the benefits for teachers and/or students for using the second definition rather than the first? What are the drawbacks?

The list of criteria for definitions indicated that when possible definitions should be hierarchical. However, students tend to prefer definitions that are **partitional** (de Villiers, 1994). A partitional definition is one that considers concepts disjoint from each other. So for example, a rectangle would be a quadrilateral with four right angles and opposite sides parallel and congruent, but consecutive sides not congruent. This last phrase does not allow for a square to also be a rectangle. We could also characterize the first definition of a trapezoid as partitional and the second definition as hierarchical. Teachers and mathematicians tend to prefer hierarchical definitions because it makes it easy to see relationships among different objects and their properties.

CONSIDERATIONS FOR TEACHING

Q19. Provide at least three possible reasons for why students might tend to prefer partitional definitions rather than hierarchical definitions.

Section 3: Students' Levels of Geometric Thinking

The van Hiele levels of geometric thinking originated in the doctoral dissertations of Pierre van Hiele and his wife Dina van Hiele-Geldof, completed in 1957 at the University of Utrecht, The Netherlands. While Pierre's dissertation dealt with school children's insight as it related to the learning of geometry, Dina's dissertation dealt with the teaching of geometry. Their research is best known for its identification of five discrete cognitive levels of thought that occur in the development of students' understanding of geometry (these levels are sometimes numbered 0-4 unlike 1-5 used here). These levels include:

Level 1: Visualization level. Students recognize a geometrical object, and they are able to name and compare objects based on their appearances.

Level 2: Description level. Students identify a figure based on its properties. The properties are seen as independent of one another. These properties are often discovered empirically.

Level 3: Informal deduction level. Students are able to analyze the interrelated properties of figures. Students also understand relationships between different figures.

Level 4: Formal logic level. Students construct deductive arguments and are able to establish interrelationships among theorems.

Level 5: Axiomatic level. Students make comparisons among different deductive systems and explore different geometries based on various postulates

(van Hiele, 1986).

Important aspects of the van Hiele theory include assumptions that: learning is a discontinuous process that involves discrete jumps from one level to the next; the levels of thinking are arranged hierarchically; what is implicitly understood at one level becomes explicitly understood at the next level; and learning involves language. The identification of levels, instead of stages or periods, suggests that the movement of a student from one level to the next is not simply a function of development over time. Rather, progression from one level to the next can be influenced by instruction and experiences with geometry. If one considers a student's van Hiele level, then the nature of tasks and activities should be adjusted accordingly. In the chart below is a description of the tasks and questions a teacher might pose to a student who is operating at each of the different levels.

van Hiele Level	Suggestions for Instruction
1. Visualization	Sorting different figures and describing shapes . View objects positioned differently as belonging to same class. Composing or decomposing shapes.

2. Description	Begin to describe different polygons using properties. Create definitions of different polygons by exploring common properties.
3. Informal Deduction	Classify different polygons using a hierarchical model. Look for generalizations and counter-examples. Create and test conjectures. Determine necessary and sufficient conditions for different polygons.
4. Formal Logic	Create conjectures and formally prove that they are always true. Analyze different proofs and determine if they are correct. Determine if different statements of the same theorem are equivalent .
5. Axiomatic	Compare and contrast properties of polygons in different geometries. Investigate what happens when an axiom of a geometry is modified. Compare and contrast different geometries.

Most students enter a high school geometry course at level 2 or below. Through instruction, teachers can often assist students in attaining level 4 at the end of the course. This is important to consider when teaching geometry at the high school or middle school level and selecting appropriate activities for students.

Tech Tip:

ENGAGING WITH CONTENT

Q20. Use the chart you created in Q14 to determine whether the following statements are sometimes true, always true, or never true.

- A. A rectangle is a parallelogram
- B. A square is a rectangle.
- C. A rectangle is a square.
- D. A cyclic quadrilateral is a parallelogram.
- E. A square is a trapezoid.
- F. A rhombus is a parallelogram.
- G. A square is a rhombus.
- H. A parallelogram is a rhombus

CONSIDERATIONS FOR TEACHING

Q21. Which van Hiele level of thinking is needed for a student to make a list of properties of different quadrilateral using a DGE sketch? What van Hiele level of think is needed for a student to classify the quadrilaterals?

Q22. How might students' uses of a DGE to explore quadrilaterals and their properties influence their geometric thinking?

Q23. Students often have difficulty determining whether statements, such as those in Q20, are always true, sometimes true or never true. Use the van Hiele levels to provide an explanation for why this may be a difficult task for students. Describe how a teacher could assist students in answering questions such as these.

Q24. Consider the topic, properties of quadrilaterals. Create one task appropriate for a student who is at each of the five different van Hiele levels.

Section 4: Analyzing Students' Geometric Thinking

The videoclip, “RightTriangle.mov” was included in chapter one to illustrate how students sometimes have difficulty recognizing geometrical objects when they look different from a common representation with which they may be familiar. View the videoclip “RightTriangle.mov” again but this time focus on identifying the van Hiele level for each student. The students are considering the following question, “Can the Isosceles Triangle Maker [a triangle that has been constructed as isosceles] make a triangle that has exactly two 45° angles?”

CONSIDERATIONS FOR TEACHING

Q25. Based on the words and actions of the student who is seated on the right, at which van Hiele level does she appear to be thinking? Provide evidence to support your claim.

Q26. Based on the words and actions of the student who is seated on the left, at which van Hiele level does she appear to be thinking? Provide evidence to support your claim.

Q27. Sometimes students who are at two different levels of geometric thinking have difficulty communicating because they associate different meanings to the same words. Do you observe any issues related to communication in this videoclip? Describe.

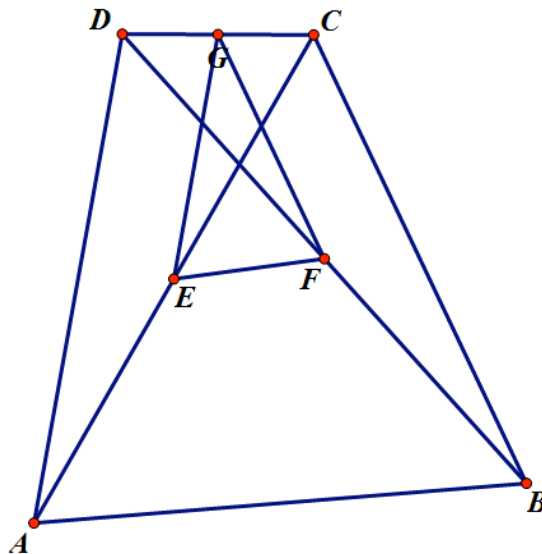
Q28. In what ways does the technology assist communication between the two students?

Q29. Describe how the teacher took into consideration the van Hiele levels of the students in the creation of the task. What might happen if the teacher and student are approaching a task at different levels?

Section 5: A Quadrilateral Exploration

It is important for students to be familiar with definitions, properties, and relations among quadrilaterals because this knowledge can assist students in solving problems that involve quadrilaterals. Consider the following situation:

Open the file “quadrilateral_exploration.gsp.” It has been created such that quadrilateral $(ABCD)$ has one pair of opposite sides congruent. The diagonals of the quadrilateral and their midpoints (points E and F) were constructed. You will also notice a triangle using the midpoints of the diagonals and the midpoint of one of the non-congruent sides (Triangle EFG) has also been constructed.



Tech Tip:

ENGAGING WITH CONTENT

Q30. What do you notice? Create at least two different interesting conjectures based on the diagram.

Q31. Select one of the two conjectures and determine whether it is never true, sometimes true, or always true.

CONSIDERATIONS FOR TEACHING

Q32. To determine whether a conjecture is always true it is likely that you constructed a mathematical argument or proof. Describe how your work with the technology was related to the construction of your argument.

Q33. Describe some ways that a teacher can assist students in using the technology to gather data that can assist them in constructing formal mathematical arguments.

Q34. Create a new problem that extends the given situation and provide a sample solution. Recall that problem-posing strategies were presented in Chapter 1.

Dragging. In this chapter and in Chapter One the technology was used in a variety of different ways. It has been used to investigate parent/child relationships, construct geometrical objects, examine relationships between constructions and dynamic behaviors, explore properties, and solve problems. In each of these cases, the way in which you used the dragging feature of the tool may have varied. Researchers who have examined students' uses of the drag feature have characterized these different dragging strategies using the following terms and descriptions.

Wandering dragging: moving a point in the plane randomly while observing relationships and behaviors.

Guided dragging: dragging points to achieve a particular shape (for example, dragging vertices of a triangle to create a scalene triangle).

Dragging test: Dragging a point of an object to see if it maintains its intended properties (for example, constructing a square and dragging a vertex to determine whether it is still a square).

Dummy locus dragging: dragging a point to preserve a particular geometrical configuration. To do so the student is dragging the point along a path, or locus, although he or she may not be aware of this path (for example, in a right triangle that has been drawn and not constructed, to maintain its right angle a student might drag a point along a path that creates a circle).

Tech Tip:

ENGAGING WITH CONTENT

Q35. Consider the quadrilateral problem presented at the beginning of Section 5. Describe the dragging strategies that you used and the purpose for using that particular type of dragging.

Q36. Were certain dragging strategies more or less useful to solving the problem? Explain.

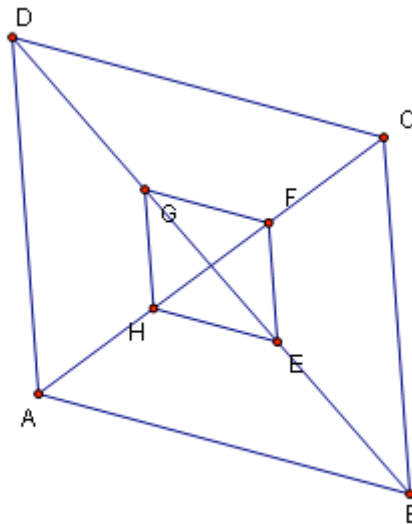
CONSIDERATIONS FOR TEACHING

Q37. Would you want to present these different dragging methods to students? Why or why not?

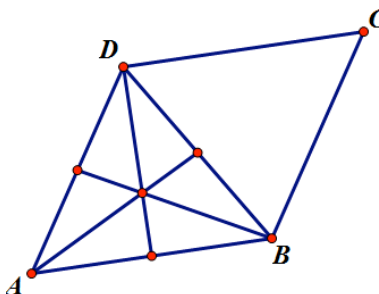
Q38. View the videoclip, "Dragging.mov" Describe how observing the ways in which this student is dragging provides insights into what he or she is thinking.

Transitioning from Conjecturing to Proving. Chapter One focused on exploration, conjecturing, and problem-posing. In high school geometry we often want students to go beyond conjecturing and transition to proving. In the situation at the beginning of this section you were presented with a quadrilateral with one pair of opposite congruent sides and a triangle that whose vertices included the midpoints of the diagonals and the midpoint of one side of the quadrilateral. You may have noticed something interesting about the triangle as you were dragging, perhaps by wandering dragging or creating a dummy locus. You might have asked yourself, “Why does this happen?” Such questions are not uncommon for students to ask also. These situations provide the perfect context for encouraging students to consider properties of geometrical objects to develop a mathematical justification that builds on what they know (the givens) to explain what they do not know and want to determine (what they want to prove). However, unlike most proofs in geometry textbooks, when students are working with a DGE the “given” and “proof” statements are usually not provided. Students must determine from the phenomena they observe in the DGE the givens and proof statements are and then develop an argument. Let’s consider another problem.

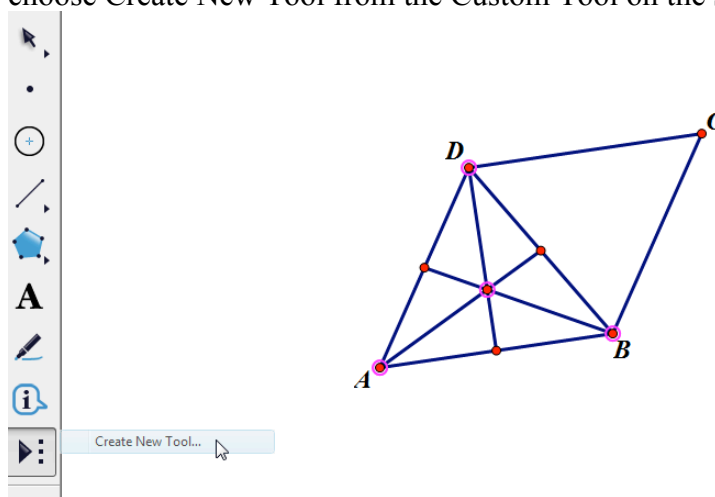
Let $ABCD$ be a parallelogram. Let E , F , G , and H be the centroids of triangles ABC , BCD , CDA , and DAB , respectively. What type of quadrilateral is $EFGH$? [Martinez-Cruz & Contreras, 2009, p. 253].



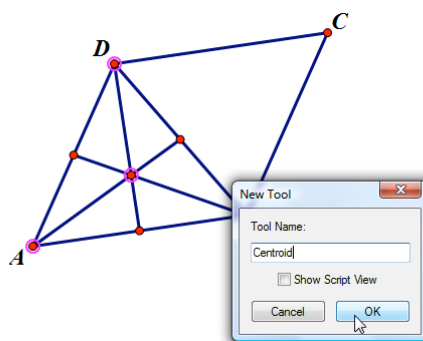
1. Begin by constructing parallelogram $ABCD$ (You may use directions in Chapter 2, Section 2.).
2. Construct diagonal BD and construct the centroid of triangle ABD (You may use directions in Chapter 1, Section 6).



3. Because you will be creating four different centroids, rather than repeat the process of constructing a centroid, most DGEs provide tools or scripts that allow you to construct the final object (centroid) without repeating the steps each time.
 - a. To create a **script or tool**: select the initial objects (vertices of the triangle) and the final object(s) created from them (centroid) and choose Create New Tool from the Custom Tool on the side menu.



A window will pop up asking you to name this script.



- b. To use this script, select this tool from the Custom Tool on the side menu and then click on each of the three vertices of the new triangle that you would like to use and its centroid will be constructed. You will not need to re-select the script tool. Simply

keep clicking on the vertices of the triangle for which you would like to find the centroid. When you have constructed all centroids, click on the selector tool to de-select the script tool.

4. Create quadrilateral EFGH by selecting those points and pressing Ctrl+L. Finally, clean up the construction by hiding unnecessary lines, line segments, and points.

Tech Tip:

ENGAGING WITH CONTENT

Q39. What type of quadrilateral is $EFGH$? How did you arrive at your conclusion?

Q40. Determine the “given(s)” and the statement “to prove” and construct a formal argument to validate the claim you made in Q36.

CONSIDERATIONS FOR TEACHING

Q41. What prerequisite knowledge would students need to know in order to construct a proof such as the one that you created in Q40?

Q42. Create four additional tasks that you could pose to students that are extensions or modifications of this problem.

Most of the tasks that you have examined in chapters 1 and 2 have been non-routine and open-ended. Tasks are sometimes described as **closed**, **open-middled**, or **open-ended** (NCTM, 1999). A closed task is one in which there is a single correct answer and a single correct solution and strategy. An open-middled task is one in which there is one correct answer but multiple correct solution strategies. An open-ended task is one in which there are multiple correct answers and multiple solution strategies.

CONSIDERATIONS FOR TEACHING

Q43. For the two tasks included in this section, determine whether they are closed, open-ended, or open-middled. Provide justifications for your decisions.

Q44. Create an open-middled or open-ended task that provides opportunities for students to apply their knowledge of properties of quadrilaterals while using a DGE. The task should include a statement of the problem and any necessary technological directions. Include a screenshot of the finished sketch and a solution to the problem.

SUGGESTED ASSIGNMENTS

H-Q1. Create a DGE file to assist students in developing a definition for cyclic quadrilateral. Describe what properties you considered in the development of the example and non-examples.

H-Q2. Construct a square, rectangle, rhombus, parallelogram, kite, and cyclic quadrilateral so that whenever a vertex is dragged the properties of the figure are preserved. Describe some of the challenges you encountered with these constructions.

H-Q3. Classify the quadrilaterals described in H-Q2 by focusing on properties of their diagonals. Create a hierarchical chart and be prepared to explain how you decided where to place each of the quadrilaterals.

H-Q4. Design a geometrical task that involves triangles or quadrilaterals and makes use of a DGE that could assist students in bridging from van Hiele level 2 to van Hiele level 3.

H-Q5. Create a quadrilateral with your DGE and label it $ABCD$. Construct the midpoints of each of the sides E , F , G , and H and create segments joining consecutive pairs of midpoints to create a new quadrilateral $EFGH$, called the midpoint quadrilateral.

- A. Explore the sketch and create at least three different conjectures.
- B. Describe the different type of dragging strategies you used to develop each of the conjectures.
- C. Select one of the conjectures and construct a formal proof.
- D. Reflect on your thinking. What is the minimum van Hiele level of geometric thinking that would be necessary for a student to successfully answer parts A, B, and C.
- E. Suppose a student drags a point so that quadrilateral $ABCD$ looks like a bowtie. How do you respond?
- F. Create a new task that could be posed to students that builds on this task.

H-Q6. In Chapter One you explored the incenter of a triangle and noted that the angle bisectors of a triangle always intersect in a single point in the interior of a triangle. Consider the case of the quadrilateral. Will the angle bisectors of a quadrilateral intersect in a single point in its interior? Explain and be prepared to present your findings.

H-Q7. A class of students was asked to complete the following proof in a high school geometry class.

Given: Parallelogram $ABCD$.

Prove: Diagonal AC bisects diagonal BD .

Consider the proofs provided by three different students and explain how you respond to each student.

STUDENT 1:

Statements	Reasons
1. Parallelogram $ABCD$ and diagonals AC and BD	1. Given
2. Label the intersection of diagonals AC and BD , point E	2. The diagonals of a parallelogram intersect in one point.
3. $AD \parallel BC$; $CD \parallel AB$	3. If a quadrilateral is a parallelogram, then its opposite sides are congruent.
4. Angle ADB is congruent to Angle CBD and Angle BCA is congruent to Angle DAC .	4. If parallel lines are cut by a transversal, then alternate interior angles are congruent.
5. Angle DEA is vertical to Angle CEB .	5. Definition of vertical angles.
6. Angle DEA is congruent to Angle CEB .	6. If two angles are vertical then they are congruent.
7. Triangle DEA is congruent to Triangle BEC .	7. If three angles of one triangle are congruent to three angles of another triangle then the two triangles are congruent.
8. $DE = EB$; $EA = EC$	8. Corresponding parts of congruent triangles are congruent.
9. AC bisects DB .	9. If two segments are divided equally then they bisect each other.

STUDENT 2:

I constructed parallelogram $ABCD$ and diagonals AC and BD . I found the midpoint of segment BD and it was also on segment AC . Therefore segment AC bisects segment BD because segment AC passes through the midpoint of segment BD .

STUDENT 3:

I constructed parallelogram $ABCD$ and diagonals AC and BD . I constructed the midpoint of segment AC and segment BD . It was the same point and I labeled the point M . I measured segment BM and segment DM and their lengths were equal. I also dragged point A and noticed that the lengths were still equal, therefore segment AC bisects segment BD .

Suggested Readings:

Mason, M. (2002). The van Hiele levels of geometric thinking. In Professional Handbook for Teachers, Geometry: Explorations and Applications. McDougall-Littell, Inc.