## Chapter 3: Congruence

## Summary: In this chapter, teachers are introduced to reflections, rotations, and translations using a DGE. Properties of each of these transformations are introduced and common student conceptions presented.

## Objectives:

Mathematical: Teachers will be able to

- State a definition for a geometric transformation.
- Understand what it means for a function to be one-to-one and onto.
- Determine whether a particular transformation is an isometry.
- Use isometries to assess whether two polygons are congruent.
- Recognize and apply properties of reflections, rotations, and translations.
- Describe differences between fixed points and identity mappings.

Technological: Teachers will be able to

- Use a DGE to apply reflections, rotations, and translations to points and polygons.
- Use the dragging feature to identity fixed points
- Use measuring tools to examine lengths of segments, measures of angles, and distances between points.

Pedagogical: Teachers will be able to

- Anticipate difficulties students may have learning geometric transformations.
- Consider particular aspects of polygons and transformations when selecting appropriate examples for students.
- Critique and modify tasks that build on students' understandings of geometric transformations.


## Prerequisites:

Vocabulary: Technology Files:
Emergency Technology Files:
Required Materials:

## Chapter 3: Congruence

## Section 1: When are two figures the same or different?

What does it mean for two geometrical figures to be congruent? Often students think about congruence as "same size, same shape" or two figures that "look the same."

## ENGAGING WITH CONTENT

Q1. Write down a definition of congruence without using a textbook or the internet.
Q2. Open the file, "Congruence.gsp." Determine if there are any pairs of congruent triangles. You can drag or measure segments and/or angles. Describe which ones are congruent and explain why you believe they are congruent.

## CONSIDERATIONS FOR TEACHING

Q3. Suppose a student working with the DGE file believes the green and orange triangles are not congruent because they do not look exactly the same. How do you respond?

You may recall from high school geometry several theorems you may have used to prove two triangles congruent. Such theorems are very useful. However, another method for proving two triangles congruent is to demonstrate that a specific type of transformation, an isometry, exists that maps one triangle to another. An isometry is a geometric transformation that preserves distances. This definition of congruence can also be applied to other figures, not just triangles. In this chapter we will explore three different geometric transformations that are isometries.

## Section 2: Geometric Transformations

In this section we begin by considering geometric transformations, in general, and their properties before examining specific types of geometric transformations.

## ENGAGING WITH CONTENT

Q4. What do you think of when you hear the words "geometric transformations"?
What you think of when you hear the term, "geometric transformation" may be related to experiences you have had through activities that involve geometric transformations. Perhaps you recall working with specific geometric transformations such as dilations, translations, reflections, shears, or rotations. However, all geometric transformations do not necessarily have specific names. Just as with functions you

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may recall specific types of functions (e.g., linear, logarithmic, quadratic, sinusoidal), but not all function have specific names.

In general, a geometric transformation is defined as a one-to-one, onto mapping of points in the plane to points in the plane. To unpack this definition, consider different components of it. First, like functions, geometric transformations have the same domains and ranges. The plane is the domain for geometric transformations. Because geometric transformations are onto, the plane is the range. Recall, that the domain of a function is the set of all input values and the range of a function is the set of all output values. If a point $A$ is mapped to a point $B$, then because we are dealing with a function, $B$ is the only point $A$ is mapped to. And because the function is one-to-one, A is the only point that maps to $B$. Point $B$ is referred to as the image of $A$ and point $A$ is referred to as the pre-image of point $B$.


For a transformation to be onto means that for all image points there exists a preimage point that can be mapped to it.


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| mapped to point $E$. This is not an onto mapping |  |
| :--- | :--- |
| from $\{A, B\}$ to $\{C, D, E\}$. | point $F$ is mapped to point $G$. This is <br> an onto mapping from $\{H, F\}$ to $\{I$, <br>  <br>  <br> $G\}$. |

In general, transformations over different sets can be defined in different ways. For example, we could define a transformation over the real number line such that for each real number $x, x \rightarrow x+1$. This algebraic description can also be represented graphically using a number line. However, because it would be difficult to represent all $x$, we can represent a subset of this transformation as shown below.

$$
0123455678991011121314151617181920
$$

In two dimensions, we could define a transformation over the Cartesian plane such that for each ordered pair $(x, y)$ such that $x \rightarrow-x$ and $y \rightarrow y$. We can represent this transformation using points in the plane.


Like the number line representation, it is not possible to show all points mapped in the plane so a subset is depicted. The arrows in this representation represent the
mapping. However, in general, the arrows are not shown and it is understood that the point without the prime is the preimage and the point with the prime is the image.

## ENGAGING WITH CONTENT

Q5. Determine whether the following mappings are geometric transformations. Explain.
A) $(x, y) \rightarrow(1,4)$
B) $x \rightarrow x^{3}$
C) $(x, y) \rightarrow(2 x+1,-3 y-6)$

Q6. Draw a diagram showing two sets, one that has 5 points and one that has 7 points. Create a mapping that is not a transformation.

Q7. Is it possible for a mapping to be onto but not one-to-one? Explain and provide an example or counterexample.

Q8. Is it possible for a mapping to be one-to-one but not onto? Explain and provide an example or counterexample.

Q9. Create an example of a mapping that is a transformation. You can use algebraic symbols or mapping diagrams.

## CONSIDERATIONS FOR TEACHING

Q10. Mapping diagrams, number lines, coordinate planes, and algebraic rules have been used to describe different transformations. Which of these representations is most meaningful to you? Explain. Which representation do you believe would be most helpful to students? Explain. If you were teaching a unit on transformations, would you want to use a one or more than one representation? Why?

## Section 3: Reflections

A Reflection is a geometric transformation where there exists a line for which all points are mapped to "mirror" images across this line of reflection of reflections.

We can use a DGE to explore properties of reflections.

1. Begin with a new sketch and create five different points.
2. Construct a line that passes through points $D$ and $E$. This is referred to as the line of reflection or a mirror line.
3. Select points $A, B$, and $C$ and reflect them over line $D E$.
4. Notice that three new points, $A^{\prime}, B^{\prime}$, and $C^{\prime}$ appear. If these points are not labeled, then use the label tool to do so.

## ENGAGING WITH CONTENT

Q11. Drag point $A$. What happens? Why?
Q12. Predict what will happen if you drag point $D$. What happens? Why?
Q13. Predict what will happen if you drag point $E$. What happens? Why?

Q14. Create segment $\mathrm{AA}^{\prime}$. What do you notice about this segment and the line of reflection?

Q15. Based on your interactions with the sketch, describe at least three different properties of reflections.

Q16. Is a reflection an isometry? Explain.

## CONSIDERATIONS FOR TEACHING

Q17. In this activity, you used the sketch to explore properties of a reflection. Contrast this approach to an approach in which all or some of the properties are presented first and then students use the technology to apply the properties to solve problems. What are the benefits of each approach?

As you were dragging points around in the sketch you may have noticed that in some cases a point $A$ and its image $A^{\prime}$ coincided. These points are often referred to as fixed points. A fixed point for a transformation is a point that is mapped to itself. For example, in algebra if you were given the function $f$, with rule $f(x)=2 x-1$ the value 1 is a fixed point for $f$ because $f(1)=1$. The input and output are the same. In geometry, when you are given a transformation that is a reflection and a point or set of points there are places where the input and the output are the same. These are fixed points.

## ENGAGING WITH CONTENT

Q18. Let's consider fixed points from an algebraic context. Describe a linear function for which there are infinite fixed points? Describe one or more linear functions for which there are zero fixed points? Describe one or more linear functions that have exactly one fixed point.

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Q19. Under a reflection, are there any fixed points? Explain.
Q20. When you were looking for fixed points using the DGE, how were you dragging? Did you use more than one dragging method? If so, was one method more helpful than another?

Using your same sketch complete the following steps:
6. Connect points $\mathrm{A}, \mathrm{B}$, and C with segments. Do the same with points $A^{\prime}, B^{\prime}$, and $C^{\prime}$. What do you notice about the two triangles?
7. Drag the vertices of the triangles. Is your observation still true?
8. What happens when you drag the line of reflection? Provide a description.

As you were dragging the line of reflection you may have noticed that at times it went through the triangles and at other times it did not. When students are presented with tasks of drawing the image of a figure under a reflection they often have more difficulty producing the image when the line of reflection does not pass through the figure.

## ENGAGING WITH CONTENT

Q21. Sketch the image of triangle $F G H$ reflected over line $D E$ in parts $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D.
A.


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## CONSIDERATIONS FOR TEACHING

Q22. Which of the tasks in the preceding question did you find easiest? Which was most difficult? Why?

Q23. How did the explorations of reflections that you conducted with the technology assist you in responding to these non-technological questions?

Q24. Sometimes young students are taught to think about reflections as flips. What properties of reflection are highlighted by thinking of them in terms of flips? What properties of reflection are not made as explicit by thinking of them as flips?

## Section 4: Rotations.

A rotation is a geometric transformation where all of the points are rotated a fixed angle measure about a single point. This angle of rotation is usually measured counterclockwise about the center of rotation in DGE programs. We can use a DGE to explore properties of rotations.

1. Begin with a new sketch and create four different points.
2. Label the points $A, B, C$, and $D$.
3. Click on point $D$ and mark it as the center of rotation.
4. Select points $A, B$, and $C$ and rotate them about point $D$ through an angle of 60 degrees.
5. Notice that three new points, $A^{\prime}, B^{\prime}$, and $C^{\prime}$ appear. If these points are not labeled, then use the label tool to do so.

## ENGAGING WITH CONTENT

Q25. Drag point $A$ and describe what happens. Explain.
Q26. Drag point $D$ and describe what happens. Explain
Q27. Create segments $A D, B D, A^{\prime} D$, and $B^{\prime} D$. What do you notice? Drag point $A$ and point $B$. Is your observation still true?

Q28. Delete the segments. Create triangle $A B C$ and triangle $A^{\prime} B^{\prime} C^{\prime}$. What do you notice?

Q29. Describe at least three different properties of rotations.

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Q30. Is a rotation an isometry? Explain.
Q31. Are there any fixed points under a rotation?

## CONSIDERATIONS FOR TEACHING

Q32. To perform a rotation, most DGEs require that you input a particular angle measure. In the above exercise an angle of 60 degrees was used. Describe how this design feature of the technology may influence student thinking about rotations.

Q33. Are there particular angles of rotation that would be more or less helpful to use with students? Explain.

Q34. In this example you rotated three points and then connected those points to create a triangle. How is rotating points different from rotating only polygons?

Q35. We started with rotating three points. Would you want to use more or fewer domain points? Explain.

A DGE file named "rotation.gsp" was created to allow the adjustment of both the angle of rotation as well as the center of rotation. Included in the sketch are several domain points and their images that can also be dragged. Open the sketch and consider the following questions.

## ENGAGING WITH CONTENT

Q36. Drag the movable point on the circle to change the angle of rotation. What changes and what remains the same? Why?

Q37. Create a circle that has as its center the center of rotation and has as its radius point one of the domain points. What other point must the circle pass through? Why?

Q38. Adjust the angle of rotation. What do you notice?
Q39. Drag one of the domain points. What do you notice?

## CONSIDERATIONS FOR TEACHING

Q40. Which properties of rotations are highlighted by your interactions with this sketch?

Q41. If you were planning to introduce rotations to students, would you use this premade sketch or have students apply a particular rotation to a set of points using their own DGE sketch. Explain the benefits and drawbacks of each approach.

Q42. In this sketch a circle was used to control the angle of rotation. A point along a line segment that behaves as a slider could also be used to control the angle of rotation [See the sketch "Rotation_Slider.gsp as an example]. What are the advantages or disadvantages to controlling the angle of rotation with a slider versus a circle?

A rotation is defined by a center of rotation and an angle of rotation. However, there are certain angles of rotation that can be applied that will map ALL points to themselves. This type of mapping is often called an identity mapping. You likely already have experience with such mappings. For example, the function $f(x)=x$ can be thought of as an identity mapping because it maps each input to an output that has the same value.

## ENGAGING WITH CONTENT

Q43. Use your DGE to apply at least two rotations to produce the identity mapping and describe them.

## CONSIDERATIONS FOR TEACHING

Q44. Students often have difficulty distinguishing between identity mappings and fixed points. Explain why they might confuse these two ideas and then create an explanation and/or examples to assist students in understanding the differences.

Q45. Many students believe that a rotation through 180 degrees is equivalent to a reflection. Is this true? Explain.

Young children can think about rotations from their own experience spinning around in circles. However, with such kinesthetic experiences the "center of rotation" is often the center of one's body or the center of a circle of friends when playing a game such as ring-around-the-rosie. Students who are beginning to learn about rotations often assume the center of rotation is at the "center" of the figure to which the rotation is being applied. However, it is important for students to coordinate the angle of rotation with the center of rotation.

## ENGAGING WITH CONTENT

Q46. For each of the following, the center of rotation, $C$, has been placed in different locations. Sketch the image of each figure under a rotation of 90 degrees.
A.

B.

C.

## CONSIDERATIONS FOR TEACHING

Q47. Which of the rotations in the preceding item were most difficult? Why? Which was easiest?

Q48. In the preceding rotation problem, a quadrilateral was used as the figure in the domain. Are there other polygons that would be easier or more difficult for students to sketch the image of under a 90 degree rotation? Are there other angles of rotation that would be easier or more difficult?

Q49. How did the explorations of rotations that you conducted with the technology assist you in responding to these non-technological questions?

Q50. What strategies did you use to create the images of the quadrilaterals?
Q51. Create a task, different from those presented in this section that would require students to apply their knowledge of rotations.

## Section 5: Translations

A translation is a geometric transformation where there is a fixed distance such that every point is translated this same distance in this same direction. This distance and direction is specified using a vector. Much like a line segment, a vector is determined by two points. The length of this line segment gives the desired distance. However, one of the points is designated as the starting point $A$ while the other is the ending point $B$. This vector $A B$ is often drawn as an arrow with the tail being at $A$ and the pointy head at $B$; the arrow visually indicates the direction

We can use a DGE to explore properties of translations.

1. Begin with a new sketch and create five different points.
2. Label the points $A, B, C, D$, and $E$.
3. Construct vector $D E$. If your DGE does not allow you to create vectors then create line segment $D E$.
4. Mark DE as the vector. The order in which you select the points is important. If you select point $D$ and then point $E$ this will create vector $D E$. If you select point $E$ and then point $D$ this will create vector $E D$. The first letter represents the starting point or tail of the vector and the second letter indicates the ending point or head of the vector.
5. Translate points $A, B$, and $C$ by vector $D E$.
6. Notice that three new points, $A^{\prime}, B^{\prime}$, and $C^{\prime}$ appear. If these points are not labeled, then use the label tool to do so.

## ENGAGING WITH CONTENT

Q52. Drag point $D$ and observe what happens. Explain.
Q53. Predict what will happen if you drag point $E$. Drag point E and state whether your prediction was consistent with your observation. Why do the points behave in the manner that you observed?

Q54. Drag point $A$. What other point moves? Why?
Q55. Create segment $A A^{\prime}, B B^{\prime}$, and segment $C C^{\prime}$. What do you notice about these segments? Explain.

Q56. Drag point D or E such that points $A, B$, and $C]$ coincide with their images $A^{\prime}$, $B^{\prime}$ and $C^{\prime}$. What do you notice about the vector? Does this sketch represent fixed points or an identity mapping? Explain?

Q57. Is a translation an isometry? Explain.
Q58. For each of the following items, sketch the image of each polygon under a translation by vector XY.


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## ENGAGING WITH CONTENT

Q59. Which of the polygons in the preceding task was most difficult to translate? Which was easiest? Why?

Q60. Does there exist a reflection, rotation or translation under which one of the points is the image of the other?

Q61. Does there exist a reflection, rotation or translation that will map one polygon onto the other? Would such a mapping have to be 1-to-1?

## CONSIDERATIONS FOR TEACHING

Q62. Students often have difficulty in reasoning about vectors. Because representations of vectors and rays are very similar students confuse these two objects. Describe how you could assist students in understanding differences between vectors and rays.

Q63. How did the explorations of translations that you conducted with the technology assist you in responding to these non-technological questions?

Q64. Open the file, "Congruence.gsp." Describe how reflections, rotations, and translations could be used to assist students in determining whether particular triangles are congruent or not. Apply the geometric transformation that you believe could be used to demonstrate to students that two triangles are congruent.

Q65. In response to the question, "What does it mean for two polygons to be congruent" a student claims, "It is not possible for any two polygons to be congruent, because only triangles can be congruent if they can be proved to be so using SAS, SSS, ASA, and AAS." How do you respond?

Q66. A student asks, "Is it possible for two sets of points to be congruent to each other?" What answer or task would you provide to this student?

## SUGGESTED ASSIGNMENTS

H-Q1. Given the figure and its image, determine what type of geometric transformation was applied. If it is (a) a translation, draw in a translation vector, (b) a rotation, identify the center of rotation and the measure of the angle of rotation, or (c) a reflection, indicate the line of reflection.

A.

B.

C.

H-Q2. For each of the following polygons, draw their pre-image under the indicated geometric transformation.

B.

Rotation of quadrilateral MNOP about point Q
through an angle with degree measure 45

©
C.

H-Q3. Compare and contrast the introduction to translations, reflections, and rotations that was presented in this chapter with the introduction that is presented at Shodor [http://www.shodor.org/interactivate/lessons/Translations/ ] using a different technology tool. Discuss the advantages and disadvantages of each approach.

H-Q4. There are many applications of geometric transformations that students can consider after they are familiar with geometric transformations. Consult the following three websites for application ideas. Explain which of the three activities you believe would be best to present to students after they are familiar with translations, reflections and rotations. Be sure to provide justifications for your decisions.

Activity 1: Tessellations. http://mathforum.org/sum95/suzanne/tess.gsp.tutorial.html
Activity 2: Create a Ferris Wheel Animation.
http://mathforum.org/dynamic/jrk/ferris_dir/
Activity 3: Creating a Kaleidoscope.
http://www.hpedsb.on.ca/ec/services/cst/elementary/math/documents/constructing_ka leidoscope.pdf

