WHAT IS A RADIAN?

Q1 All three measurements (the radius of the circle, the length of the blue segment, and the length of arc *a*) are the same. This is because the blue segment started out as a radius of the circle, and when it rolled along the circle, it measured out an arc that is the same length as it is.

Q2 1 radian $\approx 57.30°$

Q3 A semicircle has a central angle of exactly π radians. If students are not aware of that fact, they should be able to give an estimate between 3.0 and 3.3. There are 2π radians in a complete circle.

Q4 Start with the circumference formula:

$$\text{circumference} = 2\pi r$$

This means that there are 2π radius lengths in the circumference, so the angular measure is 2π radians.

WHY RADIANS?

Q5 The measurements should verify this fundamental relationship:

$$\text{arc length} = \theta r$$

The formula will work for any angle in the given range, and for any radius. It will not work when the angle units are degrees.

Students may notice some conflicts with the units. The arc length will be in centimeters, but the θr calculation will be in radians · centimeters. Leave that discrepancy for the discussion.

Q6 When degrees are converted to radians, you get this simpler formula:

$$\text{area} = \frac{\theta}{360°}\pi r^2 = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}$$

Q7 The formula works for any angle between 0 and 2π, and for any radius, but it does not work when the angle units are degrees. Again, the number values agree, but there is an apparent discrepancy with the units.

DISCUSS

The discussion questions will be more helpful if the entire class works together. Here are some suggested points.

Q8 Although radian angle measurement is useful, an angle of one radian has no great significance. The really useful angles (180°, 90°, 60°) have measures that are irrational numbers when expressed in radians, so they cannot be expressed exactly with a decimal expansion. However, we can express them as simple fraction multiples of π (π, $\pi/2$, $\pi/3$).

Q9 A radian measurement is an angle measurement, but you can just as well think of it as the ratio of the lengths of an arc and its radius. Since it is a ratio of two linear measurements, it has no units. An advantage of this concept is that it clears up the unit discrepancy that appeared in Q5 and Q7.

Q10 The number of radians in a circle is 2π, an irrational number. It is impossible to divide a circle into an integral number of radians. It is also impossible to do this with a tenth, hundredth, thousandth, or any other fraction of a radian. Instruments that measure angles (protractors, compasses, theodolites, sextants) need to have divisions that are all the same, so the angle unit must divide the circle evenly.

One obvious solution might be simply to graduate the instruments in some fraction of π radians. In fact, that is exactly what we do ($1° = \pi/180$ radians).

1. Open **Radian Measure Present.gsp.** Press the *Go* button.

As the radius segment rolls around the circle, explain that students can think of a radian as the angle that corresponds to one length of the radius being laid out along the circumference of the circle.

Q1 How long is the blue segment? [It's equal to the radius of the circle.]

Q2 What do the red ticks mark off? [Each tick marks a distance of one radius and an angle of one radian.]

2. Press *Reset;* then *1 Radian.* The animation stops after marking off one radian.

Q3 About how many degrees are there in one radian? To coax a good guess, point out the triangle formed by points *A, B,* and *C.* You can think of arc *BC* as a side of the triangle. That would make it an equilateral triangle, but one of the sides is not straight. Would that make $\angle BAC$ greater than 60° or less than 60°?

3. Press *Show Central Angle.* It will show that $\theta \approx 57.30°$.

4. Press *Semicircle.*

Q4 Count the tick marks. How many radians are there in a semicircle? [a little more than 3]

To change the angle units, choose **Edit | Preferences.**

5. Change the Angle Units to **radians.** Angle θ appears as 1π radians.

6. So a semicircle has π radians. That means that a circle must have 2π radians. Press *1 Circle* to confirm that.

Q5 But you already knew that, didn't you? What is the circumference in terms of r? [$2\pi r$] So how many times will the radius go into the circumference? [2π] So how many radians are there in a circle? [2π]

7. Press *Reset;* then press *Go.* Press *Go* again to stop the animation with θ somewhere between 0 and 2π. Press *Show Arc.*

When entering numbers that are on the screen, click the measurement itself.

8. Here is something you can do with radians, but not with degrees. Choose **Number | Calculate.** Enter $\theta \cdot r$. Compare the calculation with the measured length of the arc. Try it with several different values of θ and r.

Q6 The formula for area of a sector is $\frac{\theta}{360°}\pi r^2$. Convert the 360° to radians and simplify. What is the new formula? $\left[\frac{\theta r^2}{2}\right]$

9. Press *Show Sector.* Use the Sketchpad Calculator and enter $\theta \cdot r^2/2$. Compare the answer to the measured arc length.

Radian Measure

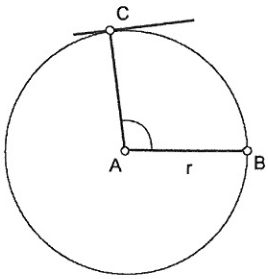
For GSP5

So you think you know angle measurement? An understanding of degrees is a valuable skill, but there are other ways to measure. Other angle units include points, grads, mils, and dekans. To make matters worse, these units may go by different names in different places. Still worse, they may have the same name, but different definitions.

Among all of these angle units, the radian holds a special place. You can use it to measure angles, of course, but radian measure also describes relationships between certain geometric objects.

WHAT IS A RADIAN?

1. Open **Radian Measure.gsp.** Press the *Go* button and watch the circle radius rotate into a tangent position and then roll around the circle.



You will use this line segment to measure a central angle of the circle. This will be the basis for defining radian measure for angles.

2. Press *Reset* and then *Home* to stop the animation and return the radius to its tangent position. Measure the radius of the circle and the length of the blue segment.

3. Press the buttons *Show Central Angle, Show Arc,* and *1 Radian.* Measure the length of arc *a.*

Q1 The central angle, $\angle BAC$, is now exactly one radian. What do you notice about your three measurements? Explain why they come out this way.

Notice that although θ is equivalent to $m\angle BAC$, it can keep increasing past 360°.

Q2 The measure of the angle, θ, is displayed in degrees. Approximately how many degrees are there in one radian?

4. Press *Semicircle.* The line segment will continue to roll until it has stepped off half of the circle.

Q3 Using the tick marks to approximate an answer, how many radians are in a semicircle? How many radians will there be in a complete circle?

5. Press the *1 Circle* button to check your last answer.

6. Choose **Edit | Preferences.** Change the Angle Units to **radians.**

Q4 The angle measurement now shows you exactly how many radians are in a circle. But you already knew that, didn't you? Write the formula for the circumference in terms of the radius. Use that along with the definition of a radian to prove that there are exactly 2π radians in a circle.

WHY RADIANS?

So far, you have not seen any good reason for using radians rather than degrees. Actually, we use radians in order to make things easier, not harder.

Do these measurements one at a time. Select one object and choose the appropriate command from the Measure menu.

7. Press *Reset* and *Go*. Press *Go* again to stop the animation before the angle makes a complete circle ($0 < \theta < 2\pi$).

8. You have a measurement for angle θ and a measurement for radius r. Use the calculator to find the product $\theta \cdot r$.

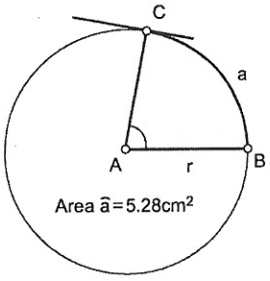
To change the radius, drag point *B*.

Q5 What is the arc length in terms of θ and r? Check your answer with different radii and different angles in the range $0 < \theta < 2\pi$. Does your formula always work? Does it work when you use degrees?

The area of a circle sector varies directly with the central angle. You probably are familiar with this formula:

$$\text{sector area} = \frac{\theta}{360°}\pi r^2$$

Q6 Rewrite the above formula using radians instead of degrees. Simplify your answer.



9. Select the arc and choose **Construct | Arc Interior | Arc Sector.** Select the sector and choose **Measure | Area.**

Q7 Using your formula from Q6, calculate the area of the sector. Does it match your measurement in all cases?

DISCUSS

Q8 When using radians, Sketchpad automatically expresses angle measurements in multiples of π. This is a common practice. Why?

Q9 It is also common practice (not used by Sketchpad) to write radian angle measurements without writing any units at all. Why is that?

Q10 In spite of the radian advantages you have seen here, degrees are more common in practical applications. What advantages do degrees have?