

Title: Concurrent Events -- An Exploration of the Points of Concurrency in a Triangle Using the *Geometer's Sketchpad*TM

Link to Outcomes:

- **Technology** Students will demonstrate the ability to use The *Geometer's Sketchpad*TM to explore the points of concurrency in any triangle.
- **Geometry** Students will use geometry vocabulary and concepts in locating the points of concurrency.
- **Problem Solving** Students will determine the specific steps needed to complete a construction.
- **Communications** Students will be able to demonstrate their understanding by discussing the results of their constructions.
- **Reasoning** Students will demonstrate their ability to reason inductively by making conjectures based upon their constructions.
- **Measurements** Students will use The *Geometer's Sketchpad*TM to obtain the size of the angle and the lengths and slopes of line segments.
- **Algebra** Students will demonstrate their ability to use ratios.

Brief Overview:

This lesson is designed to enhance student knowledge of how to effectively use the *Geometer's Sketchpad*TM to apply geometric concepts to problems that previously would have taken much longer to cover using hand-held tools; specifically, to do constructions involving the points of concurrency. Students will be able to locate the in-center, circum-center, centroid, and ortho-center for any triangle. Students will be able to construct triangles on the sketchpad. By altering the type of the triangle, students will discover the effects of this change upon the points of concurrency. Extensions are offered in both mathematical theory and real-life applications.

Grade/Level:

Grades 9 through 12 Geometry
Extensions are suggested for highly motivated students.

Duration/Length:

This lesson is designed for two to five class periods.

Prerequisite Knowledge:

Students should have some experience in the use of the *Geometer's Sketchpad*TM. Specifically, students already should be able to bisect an angle, find the midpoint of a segment, find the perpendicular bisector of a segment, drop a perpendicular from a point to a line, locate points of intersection of two lines, measure the length of segments, and construct a circle given a center and point on the circle.

Students should have also covered the following theorems via the *Sketchpad*:

- I. Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.
- II. Any point on the angle bisector of an angle is equidistant from the sides of an angle.

Students should be able to identify special segments of a triangle, i.e. median, perpendicular bisector, angle bisector, and altitude.

Students should know the following terms: acute, right, obtuse, isosceles, equilateral, midpoint of a segment, perpendicular to a line, inscribed circle, circumscribed circle, radius, center, concurrency.

Objectives:

- Students should be able to locate by construction the ortho-center, circum-center, centroid, and in-center of any triangle using the *Geometer's Sketchpad*TM
- Students should be able to use the properties of the special segments to solve further problems in geometry and extend these properties to real life situations

Materials/Resources/Printed Materials:

- *Geometer's Sketchpad*TM Software Package
- Teacher's Guide
- "Getting Started—For First Time Users of *Sketchpad*," Mary Berlin and Patricia Glasgow, *The Collected Learning Units of the 1994 Summer Institute for High School Mathematics Teachers*, Fort Meade:1994, pp. 212-213. (These pages are incorporated in this lesson.)
- Student Activity Sheets
 - a. Warm-up
 - b. Discovery Lesson #1–#4
 - c. Further exploration—The Isosceles and Equilateral Triangle
 - d. Independent Activity
 - e. Collaborative Activity
 - f. Extension and Enrichment
- One floppy diskette per student (optional)

Development/Procedures:

For both teachers and students not familiar with *Geometer's Sketchpad*TM use "Getting Started." After a comfortable level is obtained, have the students do the Warm-Up constructions on their own without teacher direction and save to diskette when completed.

Students should then begin Discovery Lessons #1–#4 while saving all constructions on their diskette (optional).

After students have completed the Discovery Lessons, they should apply their acquired knowledge by exploring the points of concurrence in an isosceles triangle and equilateral triangle using the Further Explorations worksheet. This may be used for assessment purposes.

An independent and a collaborative activity are provided for student assessment (Student Activity Sheets).

Evaluation:

- Student evaluation will be based upon the work saved on their diskette which will be turned in demonstrating their ability to perform the constructions.
- Students will also demonstrate their understanding of the concepts of the unit while exploring the points of concurrency in the isosceles triangle and the equilateral triangle (Further Exploration worksheet).
- A written performance on a set of questions designed to validate student understanding of the concepts is provided (the Student Activity Sheets).

Extension/Follow Up:

- There are three extension and enrichment worksheets that may be used to allow higher level geometry students to explore the points of concurrency in a triangle. These extensions allow the students to explore the Nine-Point circle, Euler's line, and Napoleon's theorem.

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Student Resource Sheet “Getting Started”

Getting Started—First time users of *The Geometer's Sketchpad*™

Activity One: Points

1. With the mouse on the desktop, move the arrow from icon to icon.
2. Move the arrow to the **Hand** icon and click once. The hand will appear on your screen. When the hand is near the upper right hand corner, click, hold the button down, and drag across the screen until a box appears. Release the button, and type your name.
3. Select the **Point** icon. Move to the screen and construct 5 points by clicking and releasing the mouse. Notice that the last point differs in appearance, indicating that it has been selected. Use the delete key to eliminate the point.
4. Select the **Hand** icon tool and label the points on the screen by placing the hand on the point and by clicking once. Click twice on the letter (not the point) until a box appears which will allow you to change the size of the variable shown.
5. Choose the **Arrow** icon and select a point by clicking and move the point to a new location by dragging the mouse.
6. Using the **Arrow** icon, form a box around all the points by clicking above and to the left of the first point and dragging the mouse until all the points are contained within the box. Press the **Delete** key to clear the screen.

Activity Two: Line Segments

1. Place two points on the screen while holding down the shift key. Both points will be selected. Move the arrow to the top of the screen, and pull down the **Construct** Menu. With clicker held down, move to “**Segment**” and let go. The line segment is displayed on the screen. Since the line segment is still selected, go to the **Construct** menu and choose “**Point at Midpoint.**” The midpoint of the line segment appears on the screen.
2. Place selector on **Line Segment** tool. Notice that there are 3 options. Practice drawing several line segments. In order to draw a vertical or horizontal line segment, use the shift key and the mouse, making sure you release the mouse clicker first, then the shift key.
3. Once you have numerous lines, rays, and segments on your screen go to the **Edit** menu and start “**Undoing the draw segment, rays or lines.**” Notice the lines, segments, and rays will disappear in the reverse order of construction.
4. Before ending, reselect the line segment tool.

Activity Three: Triangles and Circles

1. Place 3 points on the screen with the shift key held down. Go to the **Construct** menu and select **“Segment.”** All 3 sides of the triangle are drawn. Using the **Construct** menu, move down to **“Point at Midpoint”** observing that all the midpoints appear.
2. Move to the **Arrow** icon and click to deselect all points. With the cursor, choose 1 midpoint and its line segment. **Construct** a **“perpendicular line.”** Do the same for the other 2 sides. With only 2 perpendicular segments selected, **Construct** a **“Point at Intersection.”**
3. Select both the point at intersection and any one vertex of the triangle. **Construct** a **“circle by center + point.”**
4. Select all the Perpendicular line segments. Choose **Display** from the menu bar and **“Hide Lines.”** On the screen will be pictured a circle circumscribed around a triangle.
5. Go to the **File** menu and choose **New Sketch**. With the arrow in the upper right red box, click once.
6. Select the **Circle** icon. Moving to the screen, practice drawing circles. You might wish to draw concentric circles by selecting a common center.

Activity Four: Evaluation Exercises – Practice Drawing Circles by 3 Methods

1. Using the **Circle** icon, draw a small circle in the upper left hand corner of the screen. Label the center A.
2. In the upper right hand corner, draw a circle by selecting 2 points. Construct a circle by using **“circle by center + point.”** If the circle is not entirely on the screen, use the arrow icon to move it where it is visible. Label the center B.
3. Draw a horizontal line segment. Make sure the line segment and the left endpoint are both selected. **Construct** a **“circle by center + radius.”** Label the center C.
4. With the **Hand** icon write your name in the lower right hand corner. Choose **File** from the menu bar and select **“Print.”** Under copies select **“1,”** then move to print and click once.

Warm-Up

Do the following basic constructions using the *Geometer's Sketchpad*™.

1. Constructing the midpoint of a segment

- A. Use the segment icon to draw any segment.
- B. Select the segment.
- C. Choose *Construct* a *Point at Midpoint*.

2. Bisecting an Angle

- A. Use the segment icon to draw the sides of an angle.
- B. Select the three endpoints with the vertex in the middle.
- C. Choose *Construct* menu and select *Angle Bisector*.

3. Bisecting a segment with a perpendicular line

- A. Use the segment icon to draw a segment.
- B. Select the segment.
- C. Choose *Construct* menu and select *Point at Midpoint*.
- D. Select the midpoint and segment.
- E. Choose *Construct* menu and select *Perpendicular Line*.

4. Dropping a perpendicular from a point to a line segment

- A. Choose the segment icon to draw a segment.
- B. Choose the point icon to draw a point NOT on that line.
- C. Select both the point and the line segment.
- D. Choose *Construct* and select *Perpendicular Line*.

5. Finding the point of intersection of 3 lines

- A. Choose the segment icon to draw 3 segments intersecting at one point.
- B. Select any two of the segments.
- C. Choose *Construct* and select *Point At Intersection*.

6. Constructing a circle given a center X and a point Y on the circle

- A. Choose the point icon and draw two points.
- B. Select both of the points.
- C. *Construct segment.*
- D. Choose the hand icon and label the endpoints of the segment X and Y.
- E. Choose the arrow icon.
- F. Select in order: the center X; then Y, the point on the circle.
- G. Choose *Construct a Circle by Center+Point.*

7. Measuring the length of a segment

- A. Choose segment icon and draw any segment.
- B. Select the segment.
- C. *Measure the length.*

Point of Concurrency #1

In this worksheet, you will be investigating the properties of the angle bisectors of a triangle.

Procedure:

- A. Using the *Geometer's Sketchpad*TM, construct any triangle.
- B. Label the vertices A, B, and C.
- C. Construct the angle bisector of each angle.
- D. Locate the point of intersection.
- E. Label it "I."

This common point is called the in-center of the triangle.

Investigation:

Drag one vertex of the triangle about the screen to examine different types of triangles: acute, right, obtuse, etc.

Do the angle bisectors continue to be concurrent? _____

Where is the point of concurrency located as the triangle continues to change shape?

acute _____

right _____

obtuse _____

Use the procedure below to measure the distance from the in-center to each side of the triangle.

Procedure:

- A. Choose *Construct* a *perpendicular line* from I to one side.
- B. Choose *Construct* the *point of intersection*.
- C. Label it D.
- D. Choose *Display* and *Hide line*.
- E. Select point I and the point of intersection D.
- F. Choose *Construct* a *segment* ID.
- G. Choose *Display* and select *Color* to change the color of segment ID.
- H. Select segment ID.
- I. Choose *Measure* the *length* of segment ID.

Repeat this procedure for the other two sides of the triangle, naming the two remaining points of intersection E and F.

Drag any vertex of the triangle about the screen to examine different types of triangles—acute, right, obtuse, etc. As you change the triangle shape, what conclusion can you draw about the in-center's distance to the sides of the triangle?

Construct an inscribed circle using I as the center and D, E, or F as the point on the circle.

Construct a Circle by Center+Point

Point of Concurrency #2

In this worksheet, you will be investigating the properties of the perpendicular bisectors of the sides of a triangle.

Procedure:

- A. Using the *Geometer's Sketchpad*TM, construct any acute triangle.
- B. Label the vertices X, Y, and Z.
- C. Construct the perpendicular bisectors of each side of the triangle.
- D. Locate the point of intersection.
- E. Label it "C."

This common point is called the circum-center of the triangle.

Investigation:

Drag one vertex of the triangle about the screen to examine different types of triangles acute, right, obtuse, etc.

Do the angle bisectors continue to be concurrent?_____

Where is the point of concurrency located as the triangle changes from an acute triangle to a right triangle to an obtuse triangle?

acute_____

right_____

obtuse_____

Use the *Geometer's Sketchpad*TM to find the length of the segments from the circum-center to each vertex of the triangle.

Procedure:

- A. Change the color by choosing *Display* and *Color*.
- B. *Construct* the segments CX, CY, and CZ.
- C. Select the three perpendicular bisector lines and *hide* them.
- D. Measure the lengths of each of the segments CX, CY, and CZ.

What conclusion can you draw about the circum-center's distance to each vertex of the triangle?

Construct a circumscribed circle using C as the center and X, Y, or Z as a point on the circle.

Construct a Circle by Center+Point

Point of Concurrency #3

In this worksheet, you will be investigating the three altitudes of a triangle.

Procedure:

- A. Using the *Geometer's Sketchpad*™ construct any acute triangle.
- B. Label the vertices A, B, and C.
- C. Construct the altitudes to each side of the triangle.
- D. Locate the point of intersection of the three altitudes.
- E. Label it "O."

This common point is called the ortho-center of the triangle.

Investigation:

Drag one vertex of the triangle about the screen to examine different types of triangles: acute, right, obtuse, etc.

Do the altitudes continue to be concurrent? _____

Where is the point of concurrency located as the triangle changes from an acute triangle to a right triangle to an obtuse triangle?

acute _____

right _____

obtuse _____

State your discovery in the form of a theorem.

Point of Concurrency #4

In this worksheet, you will be investigating the properties of the medians of a triangle.

Procedure:

- A. Using the *Geometer's Sketchpad*TM construct any acute triangle.
- B. Label the vertices X, Y, and Z.
- C. Construct the midpoints A, B, and C on each side.
- D. Construct the medians XA, YB, and ZC to each side of the triangle.
- E. Locate the point of intersection of the three medians.
- F. Label it "M."

This common point is called the centroid of the triangle.

Investigation:

Drag one vertex of the triangle about the screen to examine different types of triangles acute, right, obtuse, etc.

Do the medians continue to be concurrent? _____

Where is the point of concurrency located as the triangle changes from an acute triangle to a right triangle to an obtuse triangle?

acute _____

right _____

obtuse _____

Drag one vertex of the triangle about the screen so that an acute triangle is drawn.

Use the procedure on the following page to measure the specified segments.

Procedure:

- A. Determine the length of segment XM (vertex to centroid).
- B. Determine the length of MA (centroid to other endpoint).
- C. Repeat this procedure for each median.
- D. Record your calculations in the chart below.

	XM	MA	YM	MB	ZM	MC
acute						
right						
obtuse						

Drag one vertex of the triangle about the screen so that a right triangle is shown. Repeat the procedure for a right triangle.

Drag one vertex of the triangle about the screen so that an obtuse triangle is shown. Repeat the procedure for the obtuse triangle.

What conclusion can you draw about the relationship between the distance from one endpoint of a median to the centroid and the distance between the centroid to the other endpoint of a median?

State your discovery in the form of a theorem.

Further explorations with special triangles

THE ISOSCELES TRIANGLE

Procedure:

1. Use the Sketchpad to construct an isosceles triangle.
 - a. Construct a large segment.
 - b. Select an endpoint and double click the endpoint.
 - c. Select both the segment and the other endpoint.
 - d. Choose *Transform* and *Rotate*. A dialog box will appear.
 - e. Choose *OK*.
 - f. Connect the remaining side of the triangle.
2. Construct the in-center of the triangle.
3. Label it "I."
4. Choose *Display* and *Hide* the angle bisectors.
5. Construct the circum-center of the triangle.
6. Label it "C."
7. Choose *Display* and *Hide* the perpendicular bisectors.
8. Construct the centroid of the triangle.
9. Label it "M."
10. Choose *Display* and *Hide* the medians.
11. Construct the ortho-center of the triangle.
12. Label it "O."
13. Choose *Display* and *Hide* the altitudes.

What appears to be true about the location of all four points?

State your discovery in the form of a theorem:

Describe the location of the points of concurrency in an isosceles right triangle.

The Equilateral Triangle

1. Use the sketchpad to construct an equilateral triangle.
 - a. Construct a large segment.
 - b. Select an endpoint and double click the endpoint.
 - c. Select both the segment and the other endpoint.
 - d. Choose *Transform* and *Rotate* by 60 degrees.
A dialog box will appear.
 - e. Choose *OK*.
 - f. Connect the remaining side of the triangle.
2. Construct the in-center of the triangle.
3. Label it "I."
4. Choose *Display* and *Hide* the angle bisectors.
5. Construct the circum-center of the triangle.
6. Label it "C."
7. Choose *Display* and *Hide* the perpendicular bisectors.
8. Construct the centroid of the triangle.
9. Label it "M."
10. Choose *Display* and *Hide* the medians.
11. Construct the ortho-center of the triangle.
12. Label it "O."
13. Choose *Display* and *Hide* the altitudes.

What appears to be true about the four points of concurrency?

State your discovery in the form of a theorem:

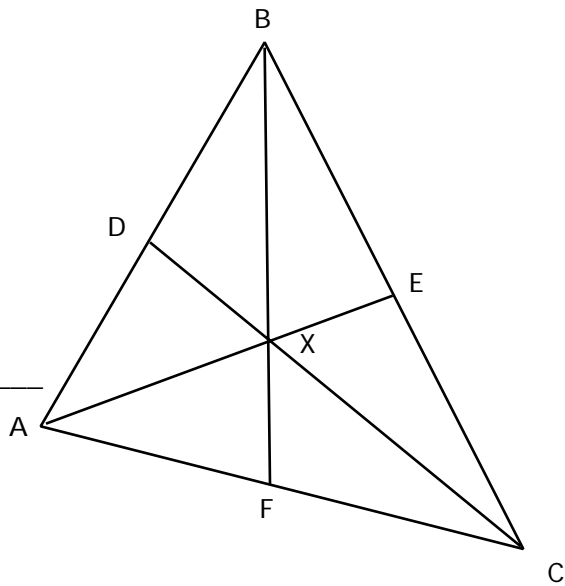
Independent Activity: Points of Concurrency

1. Describe a triangle that satisfies the following conditions:
 - a. The point where the perpendicular bisectors meet is outside the triangle.
 - b. The point where the medians meet and the point where the angle bisectors meet are on the same line.
 - c. The point where the altitudes meet is on the triangle.
 - d. The circum-center lies on the midpoint of a side of the triangle.
 - e. The ortho-center lies outside the triangle.
 - f. The centroid, circum-center, ortho-center, and in-center are all the same point.

Use the figure below for the following problems:

GIVEN: X is the centroid of Triangle ABC,
Segment BF is perpendicular to segment AC
 $AF=10$, $AD=4$, and $XF=3$

2. Find the length of FC.
3. Find of the length of BX.
4. Find the ratio of XE to XA.
5. Find the ratio of XE to EA.
6. If $BF = 6$, then $BX =$ _____
7. If $XF = 1.8$, then $BX =$ _____ and $BF =$ _____
8. If $BF = 7$, then $BX =$ _____



Collaborative Activity

Directions: Use the information that you have discovered about the four points of concurrency to solve the following problems.

1. Where is the ortho-center of a right triangle located? an isosceles triangle? an equilateral triangle?
2. Answer the same question in #1 for the centroid, in-center, and circum-center.
3. Draw a circle with a 3-inch radius and three line segments outside the circle, so that the three line segments form a triangle and are each tangent to the circle. The center of the circle is what type of point in relation to the triangle?
4. The county commissioners recently discovered that the county is in the shape of a triangle. At the last commissioners' meeting, a decision was made to build a new emergency rescue center. The commissioners need your help to determine the location of this new center so that the distance to the farthest points of the county is the same. Which point of concurrency should you locate? Compose a letter to send to the commissioners that supports your conclusion and include a diagram.
5. Mac and Tosh are stranded on an island that is in the shape of an equilateral triangle with each side of the triangle being a different beach. Mac decides that he really likes the island, would like to build a house on the island, and spend the rest of his life there. Because he is lazy, he wants to determine the ideal location to build his house so that the distance from his house to each beach is constant. Mac knows what a good geometry student you are, so he sends you a letter in a bottle asking for your assistance in determining the ideal location to build his house. Use *Sketchpad* to determine this information and write Mac a note which includes the solution to his problem.

Extension and Enrichment #1

Mystery Construction

Procedure:

1. Construct a large acute scalene triangle and label it XYZ .
2. Construct the altitude to each side of the triangle and label the points of intersection "A", "B", and "C." Label the ortho-center "O."
3. Construct the midpoint of each side of the triangle. Label the points "D", "E", and "F" so that "D" is the midpoint of segment XY and E is the midpoint of segment YZ .
4. Construct the midpoints of segment XO , segment YO and segment ZO and label the points "G", "H", and "I" respectively.
5. Construct the line segments connecting points E to G, F to H, and D to I. They should intersect in one point. Label that point "K".
6. What appears to be true about the nine points that you have constructed?
7. Test your theory using the *Sketchpad*.
8. Repeat the exercise by dragging the original triangle until it appears to be a right triangle. Verify that there is a right angle by measuring with *Sketchpad*. How many of the nine points are at the vertex of the right angle? (Watch the movement of the points CAREFULLY as the triangle approaches becoming a right triangle.)
9. Repeat the exercise, but use an equilateral triangle. What happens to some of the nine points?
10. What is the ratio of the radius of the nine-point circle to the radius of the circumscribed circle?

Extension and Enrichment #2

Euler's Line

In 1765, the Swiss mathematician, Leonhard Euler, proved that the centroid, the ortho-center, and the circum-center are all collinear. The line containing these three points is called the Euler Line. Euler also proved that the centroid of a triangle is one third the distance from the circum-center to the ortho-center.

Procedures:

1. Construct any scalene triangle.
2. Construct and name the ortho-center "O", the in-center "I", the circum-center "C", and the centroid "M" of the triangle.
3. Hide the lines leaving only the triangle and the points constructed from step #2.
4. Construct segment CM and segment MO.
5. Use the *Geometer's Sketchpad*TM to verify that the centroid, ortho-center, and circum-center are collinear by determining the slope of the segment between the points.
6. Construct segment CO.
7. Verify that the distance from the circum-center to the centroid is one third the distance from the circum-center to the ortho-center.

Extension and Enrichment #3

Centroids and the *Geometer's Sketchpad*TM

Napolean's theorem

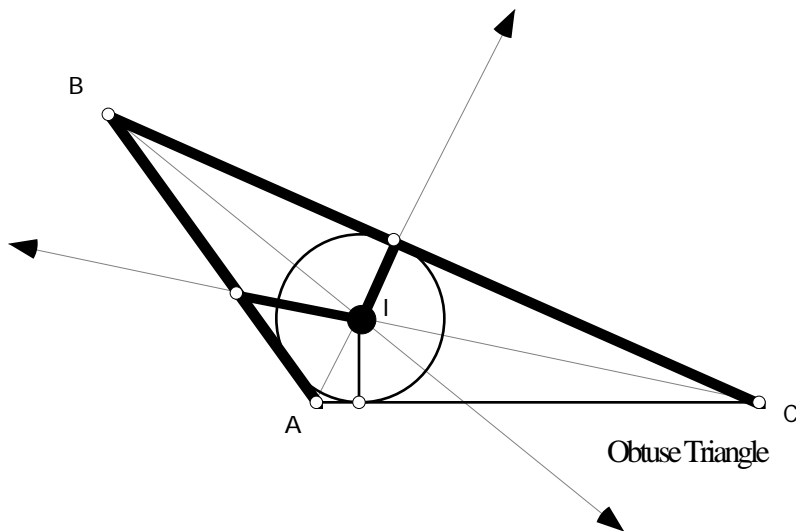
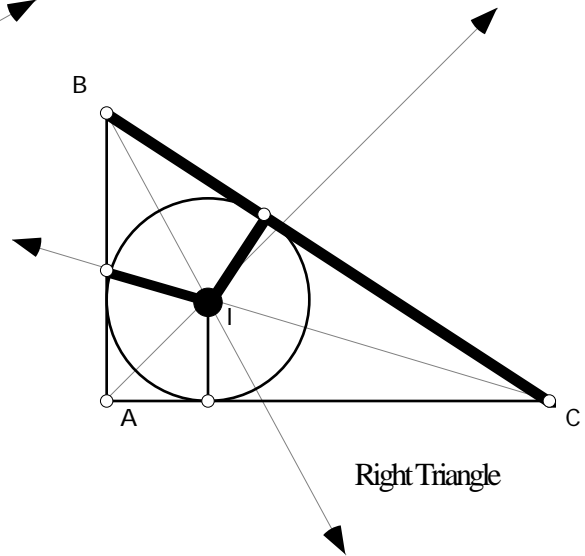
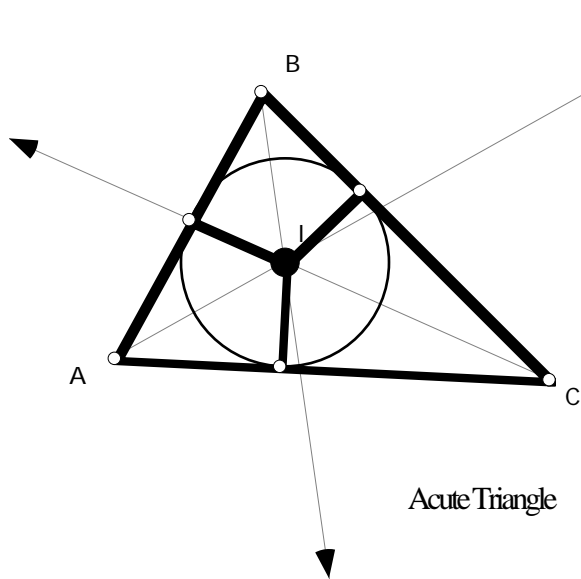
(by Napoleon Bonaparte!)

1. Draw a large scalene triangle XYZ.
2. Construct an equilateral triangle on each side of the exterior of Triangle XYZ.
3. Construct A, B, and C as the three centroids of the equilateral triangles.
4. Use the *Geometer's Sketchpad*TM to verify that Triangle ABC is equilateral.
5. Construct another equilateral triangle on each side of the interior Triangle XYZ as reflection images of the first three equilateral triangles which will overlap Triangle XYZ. (It would be highly recommended to use different colors for the triangles!)
6. Construct the centroids of these triangles and label them D, E, F.
7. Use the *Geometer's Sketchpad*TM to verify by that Triangle DEF is equilateral also.
8. Use the *Geometer's Sketchpad*TM to calculate the areas of Triangles XYZ, ABC, and DEF.
9. Verify that:

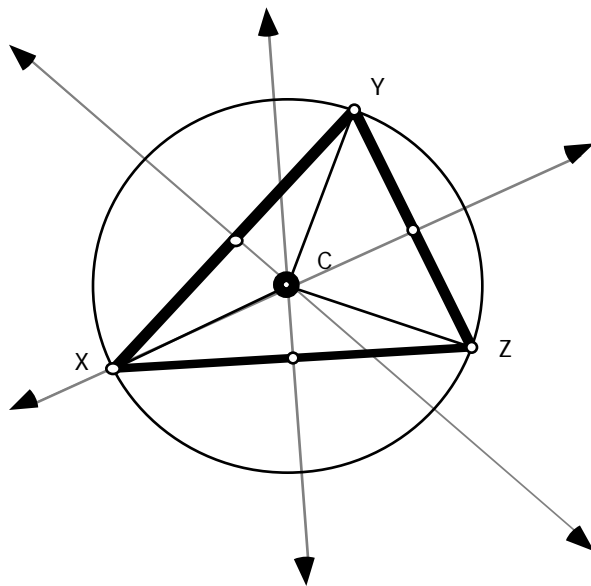
$$\text{Area of Triangle XYZ} = \text{area of Triangle ABC} - \text{area of Triangle DEF}$$

#1 Point of Concurrency for Angle Bisectors

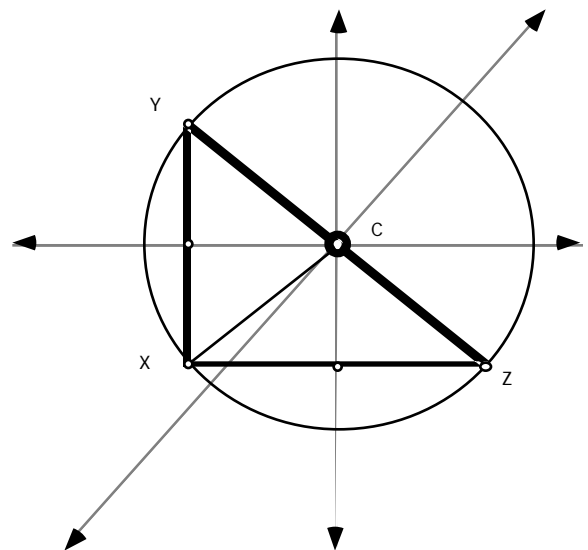
In-center



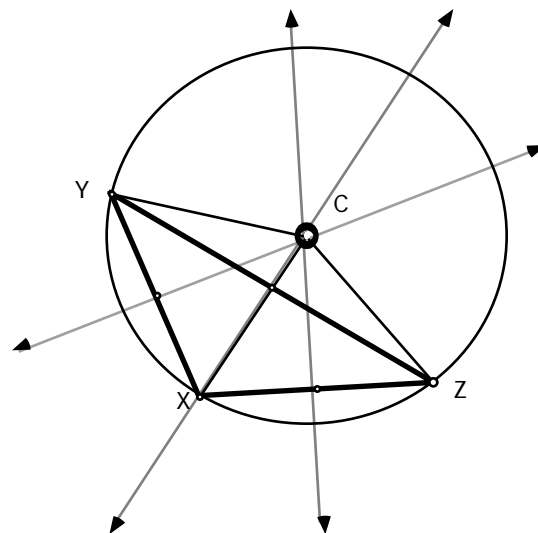
#2 Point of Concurrency for Perpendicular Bisectors Circum-center



Acute Triangle



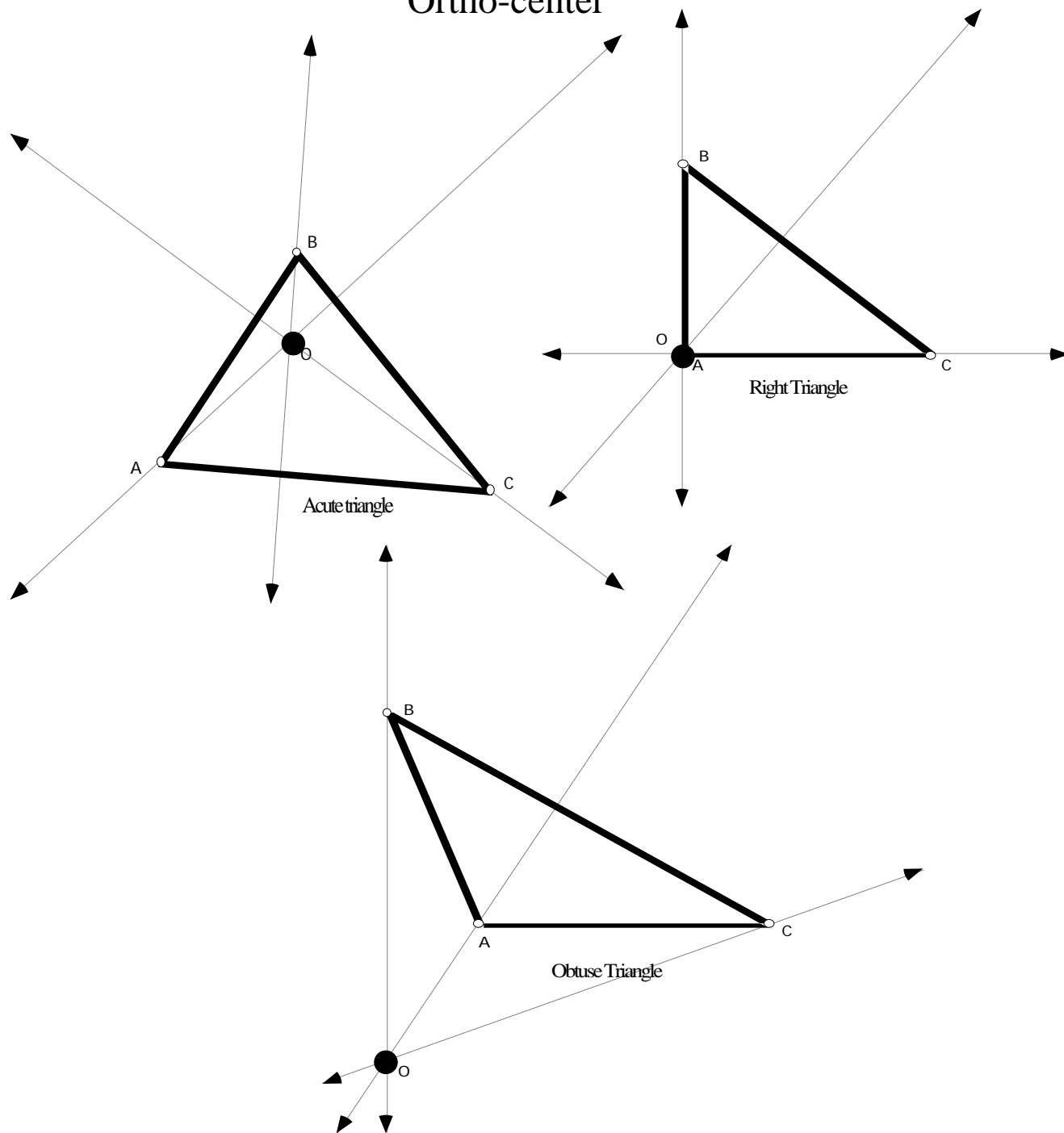
Right Triangle



Obtuse Triangle

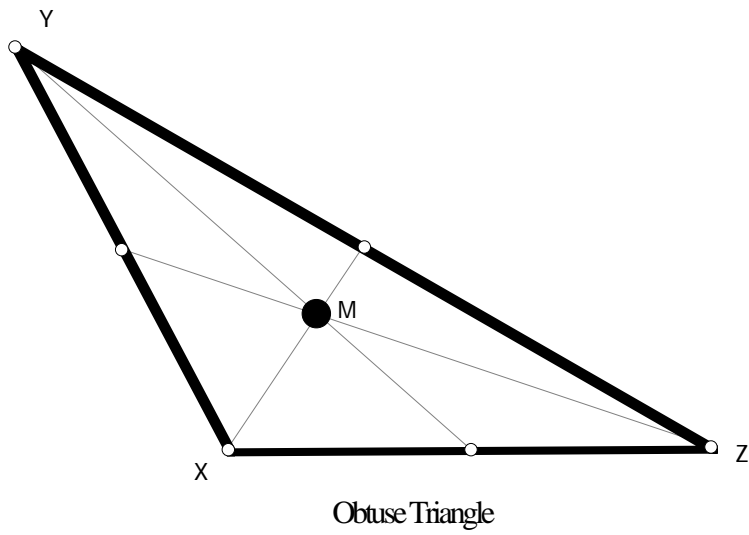
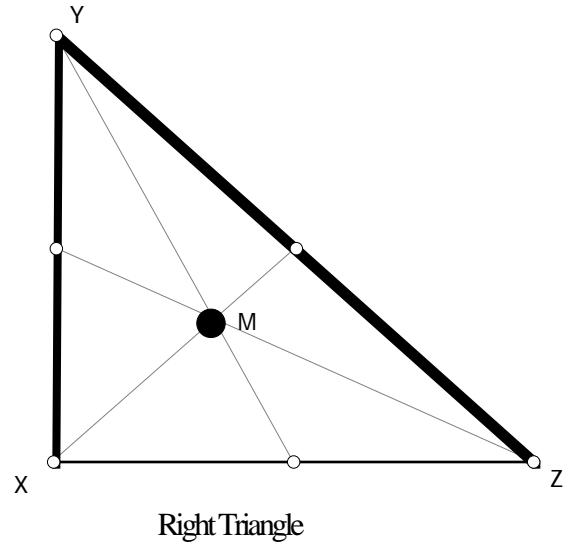
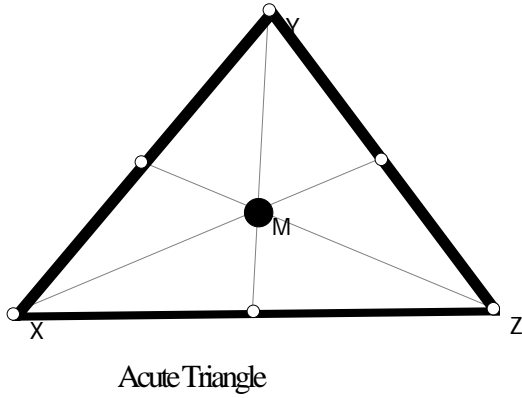
#3 Point of Concurrency for Altitudes

Ortho-center



#4 Point of Concurrency for Medians

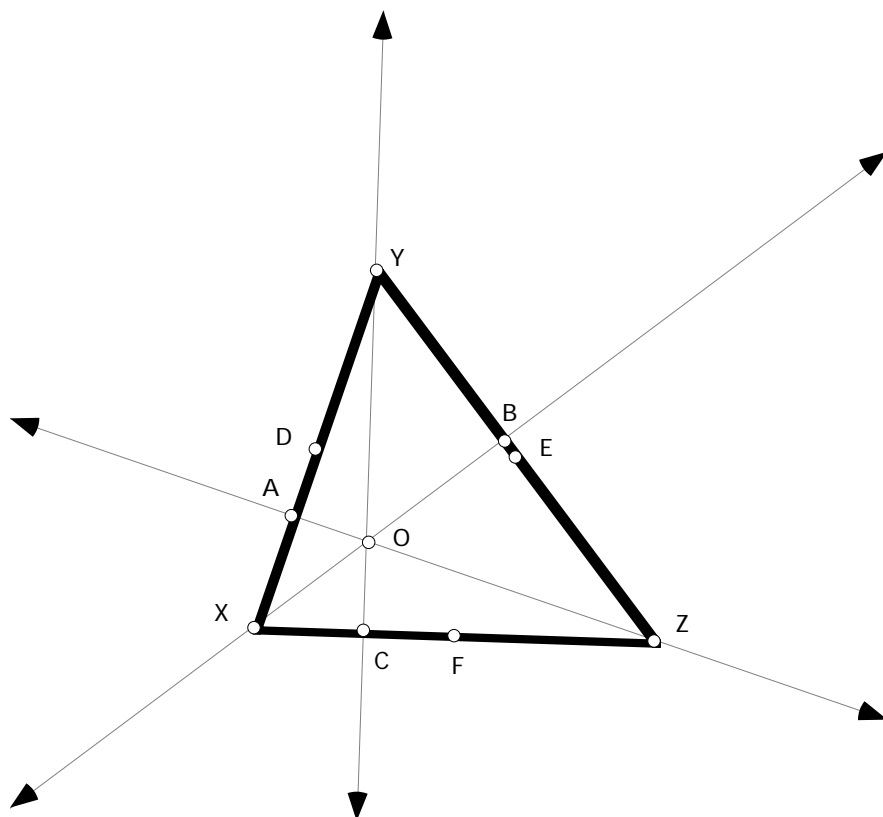
Centroid



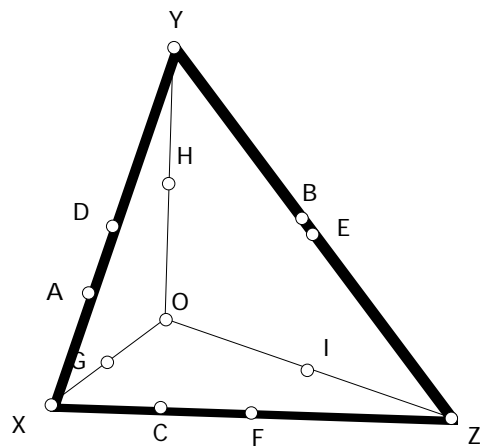
Extension and Enrichment #1

Mystery Construction

1-3

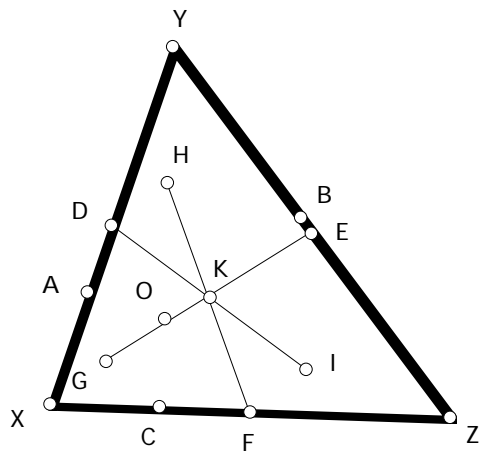


4

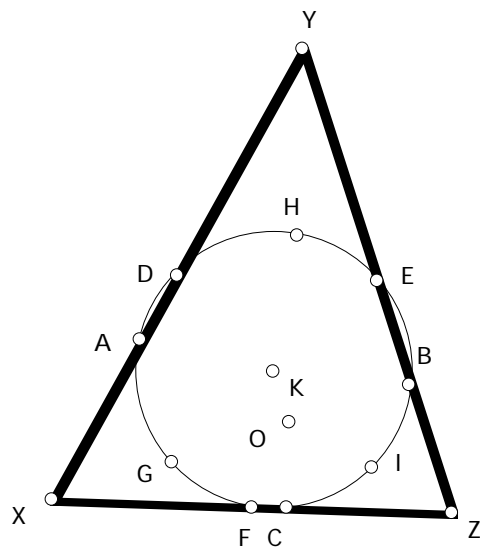


Mystery Construction (cont'd)

5

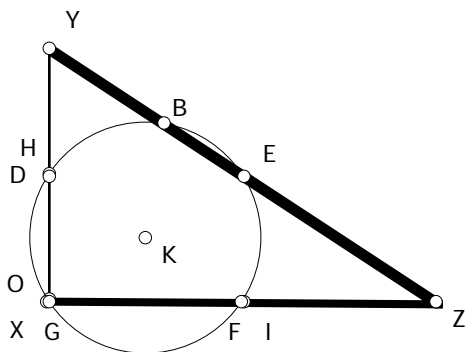


6-7

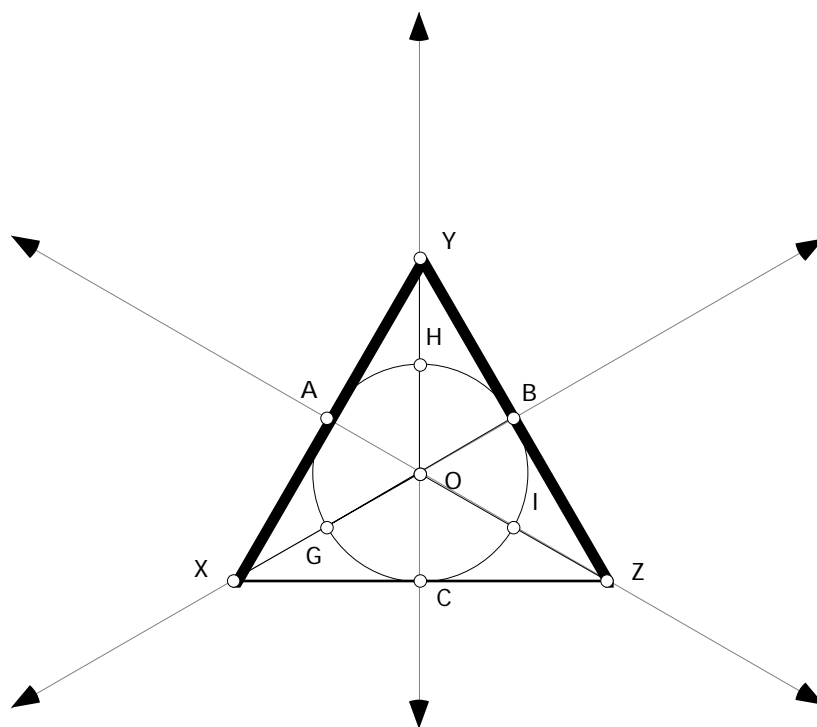


Mystery Construction (cont'd)

8



9



Extension and Enrichment #2

Solutions will vary.

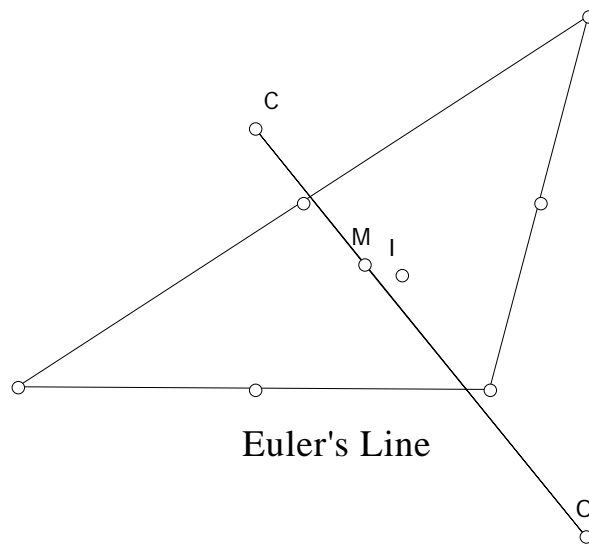
$$\text{Slope } \overline{CM} = -1.24$$

$$\text{Slope } \overline{MO} = -1.24$$

$$m \overline{CM} = 0.91 \text{ inches}$$

$$m \overline{CO} = 2.74 \text{ inches}$$

$$\frac{m \overline{CM}}{m \overline{CO}} = 0.33$$

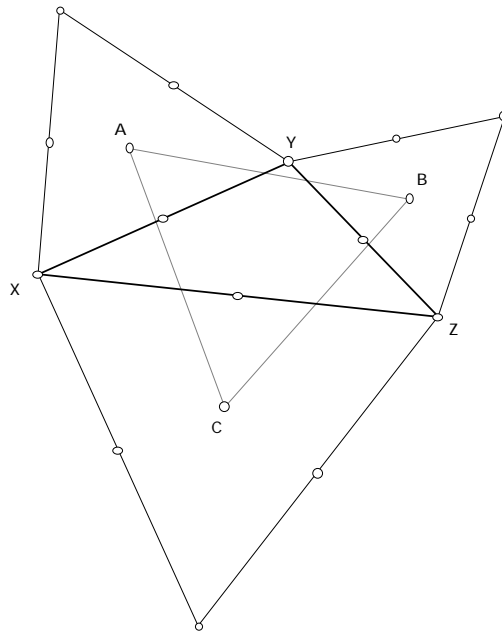


EXTENSION AND ENRICHMENT #3

Napoleon's Theorem

Solution will vary.

$m \overline{AB} = 2.13$ inches
 $m \overline{CB} = 2.13$ inches
 $m \overline{AC} = 2.13$ inches



$m \overline{AB} = 2.13$ inches
 $m \overline{CB} = 2.13$ inches
 $m \overline{AC} = 2.13$ inches
 $\text{Area } XYZ = 1.63 \text{ inches}^2$
 $\text{Area } ABC = 1.97 \text{ inches}^2$
 $\text{Area } DEF = 0.33 \text{ inches}^2$
 $(\text{Area } ABC) - (\text{Area } DEF) = 1.63 \text{ inches}^2$

